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Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) **4** (**82**) (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this is a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place. The symbol ★ precedes the title of a book or other non-periodical which is being reviewed as a whole.

References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR 19, 532: from volume 20 on, by volume and review number, as MR 20 #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

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Mathematical Reviews

Vol. 22, No. 8A August, 1961 Reviews 6661-7152

GENERAL

See also 6665, 6672, B7154, B7185, B7351, B7911.

6661:

★The international dictionary of physics and electronics. 2nd ed. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1961. v+1355 pp. $27.85.

Besides additions and revisions to the body of this dictionary—actually a glossary [1st ed. 1956; MR 19, 1206]—this edition includes (1) a new 36-page introduction outlining the connection between classical physics and modern developments and (2) an 85-page dictionary from French, German, Russian and Spanish technical terms into English.

6662:

Choquet, G. Modern mathematics and teaching. Wiadom. Mat. (2) 4, 43–57 (1960). (Polish)

Translation of a talk given at Basel and Sèvres in 1958.

6663:

Jaśkowski, S. On modernization of school mathematics. Wiadom. Mat. (2) 4, 59–71 (1960). (Polish)

HISTORY AND BIOGRAPHY

6664:

Natucci, A. Assoluta necessità che si istituiscano corsi di conferenze sulla storia della matematica e della fisica. Period. Mat. (4) 38 (1960), 228–233.

The author seeks to justify his title by exhibiting examples of historical misconceptions and anachronisms.

6665:

Seidenberg, A. The diffusion of counting practices. Univ. California Publ. Math. 3, 215–299 (1960). (2 inserts)

The purpose of the present work, according to the author, is to refute the theory that simple mathematical devices like counting were discovered again and again, being directly suggested by the uses to which they were put, and to establish his own theory that counting was diffused from one center. The author displays a vast material of counting habits, which, however, is not synthesized into a corroboration of his theory. Maybe the point will become clearer in a promised subsequent work intended to show the sacral origin of counting.

H. Freudenthal (New Haven, Conn.)

6666:

Samarkandi, Schams-ed-Din. Démonstration du V^e postulat d'Euclide par Schams-ed-Din Samarkandi. Traduction de l'ouvrage Aschkâl-üt-teessis de Samarkandi. Translated by Hâmid Dilgan. Rev. Hist. Sci. Appl. 13 (1960), 191–196.

6667:

Oettel, H. Giovanni Ceva. Ein Beitrag zur Geschichte der Mathematik in Italien um 1700. Math. Naturwiss. Unterricht 13 (1960/61), 257–260.

6668:

Ore, Oystein. Pascal and the invention of probability theory. Amer. Math. Monthly 67 (1960), 409–419.

The author indicates that the popular historical account, in which Chevalier de Méré is described as a gambler who proposed two problems to Pascal which were supposedly based on his gambling experience and whose solution initiated probability theory, is distorted. It is suggested that de Méré could better be classified as an amateur mathematician, that these problems had been of interest for many years, and that more than is generally supposed was known of probability theory at the time of Pascal.

H. Chernoff (Stanford, Calif.)

6669:

Lodge, Oliver. ★Pioneers of science. Dover Publications, Inc., New York, 1960. xiii+404 pp. $1.50.

Unaltered republication of last (corrected) edition [Macmillan, London, 1926]. A non-technical illustrated history of astronomy and its principal protagonists, from Copernicus through the late 19th century.

6670:

Truesdell, C. A program toward rediscovering the rational mechanics of the Age of Reason. Arch. Hist. Exact Sci. 1, no. 1, 3–36 (1960).

The author, convinced that the history of mechanics between Newton and Lagrange has been unduly neglected, publishes an outline of a program for the study of it. The contents may be summarized in the following list of titles of paragraphs. 1. A brief survey of the contents of Newton's *Principia*, in which both the merits and the shortcomings of the work are described in a realistic way. 2. Scientific methods in the Age of Reason are discussed, in particular Newton's *Regulae philosophandi*. 3. The contributions of James Bernoulli to rational mechanics. 4. Early efforts to formulate the principles of mechanics. 5–6. Vibrating systems and fluid mechanics before the

discovery of the equations of motion. 7. *Differential equations of motion*, as published by John Bernoulli and d'Alembert in 1743. 8. In 1750 Euler applied the principle of linear momentum to mechanical systems of all kinds. 9. At the same time the general equations of motion of a rigid body about its center of gravity were obtained. 10. Now problems of vibratory and wave motion could be solved. 11. John Bernoulli was the first to deal with the motion of a deformable body. Euler created the general concept of internal pressure. 12. The laws of elasticity and the concept of shear stress. 13. The principle of moment of momentum. 14. The invariance of the laws of mechanics. Lagrange's *Mécanique analytique* (1788). 15. Retrospect: experience, theory and experiment in the Age of Reason.

E. J. Dijksterhuis (Bilthoven)

6671:

Grigorian, A. T. Les travaux sur la Mécanique non-euclidienne en Russie. Scientia (6) 54 (1960), 347–350.

Expository history.

6672:

★Математика в СССР за сорок лет: 1917–1957 [Forty years of mathematics in the USSR: 1917–1957]. Vol. I: Survey articles. Vol. II: Biobibliography. Edited by A. G. Kuroš (ed.-in-chief), V. I. Bityuckov, V. G. Boltyanskiĭ, E. B. Dynkin, G. E. Šilov, A. P. Yuškevič. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. Vol. I: 1002 pp. 43.80 r. Vol. II: 819 pp. 44.70 r.

This compendium displaces the previous 15- and 30-year indexes [1932, 1948] of Soviet mathematics since 1917. It is intended to cover only mathematics proper; "applied" articles were included in the bibliography only to the extent that they were judged important to "pure" mathematics. Also generally excluded are methodology and didactic literature. The bibliography contains withal some 22,000 items by 3600 authors; this includes the previous 30-year bibliography which, by way of sharp contrast, contained 7000 items by 1300 authors. In contrast also to the previous indexes, the bibliography is (understandably) no longer arranged by subject matter but simply by author. For over half the authors listed, a biographical sketch is also provided: full given name and patronym (items of some interest), birthdate and place, universities attended, positions held, honors received; also references to pertinent biographical sketches in the Greater Soviet Encyclopedia and in Russian mathematical journals as well as cross-references to all mentions of the author in the survey articles of vol. I.

Vol. I covers essentially the decade 1947–57, the previous 30 years having been dealt with in the previous indexes. The articles are generally well-supplied with bibliographical references, including foreign literature to the extent that it bears on the domestic. Following are its contents:

S. A. Yanovskaya (in collaboration with S. I. Adyan, Z. I. Kozlova, A. V. Kuznecov, A. A. Lyapunov, V. A. Uspenskiĭ): Mathematical logic and foundations of mathematics.

Yu. V. Linnik: Number theory.

V. M. Gluškov and A. G. Kuroš: General algebra.

D. K. Faddeev: Theory of fields and polynomials.

E. B. Dynkin: Linear algebra.

E. B. Dynkin: Theory of Lie groups.

P. S. Aleksandrov and V. G. Boltyanskiĭ: Topology.

S. M. Lozinskiĭ and I. P. Natanson: Metric and constructive theory of functions of a real variable.

Theory of functions of a complex variable.—A. O. Gel'fond: Introduction. S. N. Mergelyan: Approximations to functions of a complex variable. M. A. Evgrafov: Interpolation of entire functions. G. C. Tumarkin and S. Ya. Havinson: Power series and their generalizations. Monogeneity problems. Boundary properties. I. E. Bazilevič: Geometric function theory. L. I. Volkovyskiĭ: Riemann surfaces. B. V. Šabat: Generalizations and analogues of the theory of analytic functions. B. A. Fuks: Functions of several complex variables. F. D. Gahov and B. V. Hvedelidze: Boundary value problems in the theory of analytic functions of a complex variable.

V. V. Nemyckiĭ: Ordinary differential equations.

M. I. Višik, A. D. Myškis and O. A. Oleĭnik: Partial differential equations.

L. A. Lyusternik: Calculus of variations.

S. G. Mihlin: Linear integral equations.

M. A. Krasnosel'skiĭ, M. A. Naĭmark and G. E. Šilov: Functional analysis.

A. N. Kolmogorov: Theory of probability.

I. I. Gihman and B. V. Gnedenko: Mathematical statistics.

M. K. Gavurin and L. V. Kantorovič (in collaboration with M. Š. Birman, V. I. Krylov, A. N. Baluev, G. Š. Rubinšteĭn and K. E. Černin): Approximate and numerical methods.

A. A. Lyapunov: Mathematical studies connected with the operation of computing machines.

M. R. Šura-Bura: Programming.

S. V. Bahvalov: Nomography.

N. F. Četveruhin: Descriptive geometry.

A. M. Vasil'ev, A. P. Norden and S. P. Finikov: Differential geometry.

N. V. Efimov: Geometry "in the large".

A. P. Yuškevič: History of mathematics.

6673:

Kuratowski, K. Ten years of the Institute of Mathematics. Wiadom. Mat. (2) 3, 199–216 (1960). (Polish)

6674:

Ważewski, T. The role of the Institute of Mathematics with respect to individual mathematical centers in Poland. Wiadom. Mat. (2) 3, 217–221 (1960). (Polish)

6675:

Borsuk, K. On the achievements of Prof. Dr. Kazimierz Kuratowski in the realm of topology. Wiadom. Mat. (2) 3, 231–237 (1960). (Polish)

6676:

Marczewski, E. On the papers of Kazimierz Kuratowski in set theory and measure theory. Wiadom. Mat. (2) 3, 239–244 (1960). (Polish)

6677:

List of papers by K. Kuratowski published during 1918–1958. Wiadom. Mat. (2) 3, 245–250 (1960). (Polish)

6678:

Mitropol'skiĭ, Yu. A.; Tyablikov, S. V. Nikolaĭ Nikolaevič Bogolyubov (on the occasion of his fiftieth birthday). Uspehi Fiz. Nauk **69** (1959), 159–164 (Russian); translated as Soviet Physics. Uspekhi **2**, 765–770.

6679:

Leray, J. The works of Julius Paul Schauder. Wiadom. Mat. (2) **3**, 13–19 (1959). (Polish)

6680:

Works of Professor Masatsugu Tsuji. Japan. J. Math. **29** (1959), 185–189.

A list of 14 books and 138 papers. This volume of Japan J. Math. is dedicated to Professor Tsuji and includes a photograph.

6681:

Bremmer, H. The scientific work of Balthasar van der Pol. Philips Tech. Rev. **22** (1960/61), 36–52.

A comprehensive survey, with many references. (A full bibliography of Van der Pol's scientific papers, by C. J. Bouwkamp, is promised for a future issue.) The main sections are: Propagation of radio waves; Non-linear circuits: relaxation oscillations; Transient phenomena and operational calculus.

6682:

Knaster B. Zygmunt Janiszewski (on the 40th anniversary of his death). Wiadom. Mat. (2) **4**, 1–9 (1960). (1 plate) (Polish)

6683:

Zygmund, A. Józef Marcinkiewicz. Wiadom. Mat. (2) **4**, 11–41 (1960). (1 plate) (Polish)

6684:

Nagell, Trygve. Anders Wiman in Memoriam. Acta Math. **103** (1960), i–vi.

6685:

Blum, J. R. Werner Gautschi, 1927–1959. Ann. Math. Statist. **31** (1960), 557.

6686:

Takács, L. Charles Jordan, 1871–1959. Ann. Math. Statist. **32** (1961), 1–11.

6687:

William J. Turanski, 1925–1960. Comm. ACM **3** (1960), A14.

6688:

Perron, Oskar. Leopold Fejér: 9.2.1880–16.10.1959. Bayer. Akad. Wiss. Jbuch. **1960**, 169–172. (1 plate)

6689:

Tietze, Heinrich. Erhard Schmidt: 13.1.1876–6.12.1959. Bayer. Akad. Wiss. Jbuch. **1960**, 176–177. (1 plate)

LOGIC AND FOUNDATIONS

See also B7775.

6690:

Mittelstaedt, P. Über die Gültigkeit der Logik in der Natur. Naturwissenschaften **47** (1960), 385–391.

6691:

Carnap, Rudolf. ★Einführung in die symbolische Logik, mit besonderer Berücksichtigung ihrer Anwendungen. 2te neubearbeitete und erweiterte Aufl. Springer-Verlag, Vienna, 1960. xii + 241 pp. $6.65.

An English version of the present edition was published in 1959 [Dover, New York; MR **21** #2578].

6692:

Łukasiewicz, Jan. ★Elementy logiki matematycznej [Elements of mathematical logic]. 2nd ed. Państwowe Wydawnictwo Naukowe, Warsaw, 1958. 99 pp. zł. 20.00.

This book first appeared in 1929 as a set of mimeographed notes edited by the Association of the Students of Mathematics and Physics at the University of Warsaw and contained the results of the author concerning the two-valued propositional calculus. The second edition, prepared by J. Słupecki, differs imperceptibly from the first one except in § 7, which contains a proof of the completeness theorem for the propositional calculus. This proof has been changed according to the later modifications given by the author. Moreover, the last part, "On the reasoning in natural sciences", has been omitted in the second edition.

The book is excellent from the didactic point of view. It is divided into five parts. Part I contains a short historical introduction concerning the development of mathematical logic. In Part II a system of the two-valued propositional calculus is described and formal proofs for some derivable formulas are presented. Note that formulas are written without parentheses, using the symbolism of Łukasiewicz. In Part III the author deals with the consistency, the independence of the axioms and the completeness theorem for the system of propositional calculus considered. Part IV deals with the propositional calculus with quantifiers and Part V contains the theory of the Aristotelean syllogism formalised on the basis of the propositional calculus with quantifiers.

H. Rasiowa (Warsaw)

6693:

Rose, Alan. Sur un ensemble indépendant de foncteurs primitifs pour le calcul propositionnel, lequel constitue son propre dual. C. R. Acad. Sci. Paris **250** (1960), 4089–4091.

It is shown that the ternary propositional connective G, where $GPQR$ has the same truth-table as $AAKPQKPNRKQNR$, together with the logical constants t and f, form a complete self-dual set of independent

connectives for the two-valued propositional calculus. The same properties are established for the system in which G is replaced by H, where $HPQR$ has the same truth-table as $AAKPNQKPRKNQR$. The connective G, together with the universal and existential quantifiers, are shown to form a complete self-dual set of independent connectives for the two-valued erweiterter Aussagenkalkül. For the origin and definition of these notions, cf. Alan Rose, J. Symb. Logic **18** (1953), 63–65 [MR **14**, 936].

G. F. Rose (Pacific Palisades, Calif.)

6694:

Rose, Alan. Nouvelle formalisation du calcul propositionnel bivalent dont les foncteurs primitifs forment un ensemble qui constitue son propre dual. C. R. Acad. Sci. Paris **250** (1960), 4246–4248.

A formalization is obtained for the two-valued propositional calculus with primitives H, t and f [cf. preceding review]. This is accomplished by means of four independent axiom schemata and a detachment rule, in such a way that the dual of each axiom and rule is easily derivable.

G. F. Rose (Pacific Palisades, Calif.)

6695:

Mleziva, Miroslav. Die Unabhängigkeit des Axiomensystems des Aussagenkalküls von Hermes und Scholz. Časopis Pěst. Mat. **84** (1959), 454–460. (Czech and Russian summaries)

This paper establishes the independence of Hermes and Scholz's fifteen axioms for the two-valued propositional calculus [H. Hermes and H. Scholz, *Mathematische Logik*, Enzykl. math. Wiss., Bd. I 1, Heft 1, Teil I, Teubner, Leipzig, 1952; MR **16**, 435]. The independence of each axiom is proved by means of a normal matrix with a single designated value.

G. F. Rose (Pacific Palisades, Calif.)

6696:

Salomaa, Arto. On the composition of functions of several variables ranging over a finite set. Ann. Univ. Turku. Ser. A I **41** (1960), 48 pp.

This paper is concerned mainly with the determination of Sheffer functions. The author shows that a function $f(x_1, \cdots, x_n)$ which generates all j-place functions is a Sheffer function, the theorem being proved first for the case $j=1$. If $n=2$ he also shows that f cannot be associative. The result for the case $j=1$ leads to an improvement in a method, given earlier in the paper, for determining whether a function is a Sheffer function. A number of particular Sheffer functions are constructed, and further results in this direction are to be published. The paper concludes with a proof that if $f(x, y)$ generates the symmetric groups $S_n (n \geq 3)$ then $f(x, y)$ is a Sheffer function, and a discussion of some problems suggested by this result. A lower bound for the number of Sheffer functions of n-valued logic is obtained, the value of the bound in the case $n=4$ being 1391616.

A. Rose (Nottingham)

6697:

Moh, Shaw-kwei. Modal systems with a finite number of modalities. Sci. Sinica **7** (1958), 388–412.

This is an English version of a paper previously published in Chinese [Acta Math. Sinica **7** (1957), 1–27; MR **21** #3].

A. Rose (Nottingham)

6698:

Yonemitsu, Naoto. A note on modal systems, von Wright's M and Lewis's S1. Mem. Osaka Univ. Lib. Arts Ed. Ser. B **4** (1955), 45.

The author shows that if t is any fixed tautology and the axiom $\sim \Diamond \sim \sim \Diamond \sim t$ is added to those of the Lewis system S1 then the resulting formal system is equivalent to the system M of von Wright. The proof makes use of the equivalence of von Wright's system and a system derived from S2, but the equivalence is not proved here; see review below.

A. Rose (Nottingham)

6699:

Yonemitsu, Naoto. A note on modal systems. II. Mem. Osaka Univ. Lib. Arts Ed. Ser. B **6** (1957), 9–10.

In the previous paper reviewed above, the author made use of the equivalence of von Wright's system M and the system obtained from the Lewis calculus S2 by adding the axiom $\sim \Diamond \sim (p \prec p)$. He now establishes this equivalence and shows also that these calculi are equivalent to that obtained from S2 by adding the axiom $\Diamond (p \sim p) \cdot \prec \cdot p \sim p$. The paper concludes with a brief discussion of applications of the above results to M, S2, S3 and S4.

A. Rose (Nottingham)

6700:

Anderson, Alan Ross; Belnap, Nuel D., Jr. Modalities in Ackermann's "rigorous implication". J. Symb. Logic **24** (1959), 107–111.

The authors show that the structure of modalities in Ackermann's system of rigorous implication (strenge Implikation) [same J. **21** (1956), 113–128; MR **18**, 270] is identical with that in the Lewis system S4 and that modalities may be defined with the help of rigorous implication.

A. Rose (Nottingham)

6701:

Bergmann, Gustav. The philosophical significance of modal logic. Mind **69** (1960), 466–485.

The author presents and defends the thesis that modal sentential calculi (MSC), as conceived by C. I. Lewis [*Survey of symbolic logic*, Univ. of Calif. Press, Berkeley, Calif., 1918; chapter 5] and some of his followers, are of no utility in the philosophy of logic, and a fortiori are of no utility in philosophical analysis, even though MSC have significance in logico-mathematical research. In addition, he defends the thesis that "... the explication of one of the several philosophical uses of modal words is the main task of the philosophy of logic."

More specifically, the author considers "axiomatic systems" (AS), e.g., an axiomatization of Euclidean geometry, where neither the axioms nor the theorems are logical truths and "pseudoaxiomatic systems" (PAS), e.g., the sentential calculus (SC) of Russell and Whitehead [*Principia mathematica*, Univ. Press, Cambridge, England, 1925–1927] where both the axioms and the theorems are logical truths. He shows two interesting similarities and differences between an AS and a PAS. He dismisses any philosophical significance to a PAS since it assumes logical truth and tends to breed certainty of the Cartesian nature. Finally, he defends the proposition that each MSC is both a PAS and a useless "standardization" of sorts. [By way of criticism, the reviewer observes that the author says,

"I believe that while some calculi are of philosophical use, some others are not. Among the latter I count the several MLC [sic]; among the former, SC." However, the author not only uses the SC of Russell and Whitehead as an example of a PAS, but he says, "... an axiomatic system, whether pseudo or genuine, never explicates any of its primitives." In spite of this the paper is worthy of being read by logicians since it contains an abundance of suggestive thoughts.}
A. A. Mullin (Urbana, Ill.)

6702:

Umezawa, Toshio. On intermediate many-valued logics. J. Math. Soc. Japan **11** (1959), 116–128.

The author defines an intermediate many-valued logic to be one in which every intuitionistically provable formula is true. He first obtains a rather long condition which is sufficient for a propositional calculus to be intermediate and then shows that there exists an enumerably infinite set $\{L_1, L_2, \cdots\}$ of intermediate logics such that no logic L_i is a sub-logic of any logic L_j. It is also shown that there is a sequence of intermediate logics of order-type ω^{ω^ω} such that each logic is a sub-logic of the preceding ones. The author then applies his results to the predicate calculus and obtains further theorems, similar to those which he obtained for the propositional calculus.
A. Rose (Nottingham)

6703:

Umezawa, Toshio. On some properties of intermediate logics. Proc. Japan Acad. **35** (1959), 575–577.

Consider the following sequents:

$$K^\circ: \quad \to \neg\neg\forall x(A(x) \lor \neg A(x))$$
$$MK^\circ: \quad \to \neg\neg\forall x A(x),\ \neg\neg\exists x \neg A(x)$$
$$FG: \quad \forall x A(x) \supset \exists y B(y) \to \exists x \exists y (A(x) \supset B(y)).$$

The system obtained from Gentzen's LJ by adding a sequent X as axiom scheme is denoted by LX and its provable formulas are called X-provable. It is then proved that LK° $[LMK^\circ]$ is minimal in the class of predicate logics LX such that: if $\Gamma \to E$ $[\Gamma \to \Delta]$ is K-provable (i.e., in Gentzen's LK) then $\neg\neg\Gamma \to \neg\neg E$ $[\neg\neg\Gamma \to \neg\neg\Delta]$ is X-provable. A sequent Y is said to be decomposed into $Z_1, \cdots, Z_n$ in LX if and only if the Z_i are (X, Y)-provable and Y is $(X, Z_1, \cdots, Z_n)$-provable. It is shown that in LFG any sequent can be decomposed into finitely many sequents with empty antecedents and succedents consisting of one formula in Skolem normal form not containing $\lor$.
B. van Rootselaar (Amsterdam)

6704:

Umezawa, Toshio. On logics intermediate between intuitionistic and classical predicate logic. J. Symb. Logic **24** (1959), 141–153.

On the basis of Gentzen's LJ the author considers systems of predicate calculus intermediate between LJ and Gentzen's LK, by adding to LJ a K-provable sequent X (which is not J-provable) as axiom scheme, so as to obtain the intermediate logic LX. The object is to settle the problems of inclusion for the intermediate logics. Previous results by the author for the propositional calculus [same J. **24** (1959), 20–36; MR **22** #4634] are shown to carry over to the predicate calculus. New results are established for calculi LX, where X is a sequent containing quantifiers. The logics LK° and LMK° [see preceding review] are interesting because they have certain minimality properties. The author draws up two diagrams showing the inclusions he has established; evidently this is a rather complicated matter and there remains a number of problems in this respect. Non-inclusion is systematically shown by using suitable Brouwer algebras. Several non-inclusions of course also follow from known unprovability results [cf., e.g., A. Heyting, ibid. **11** (1946), 119–121; MR **8**, 306]. There is one complete result: There are seven intermediate logics for sequents of the form $\to Y \lor Z$; $\to \neg\neg(Y \lor Z)$; $\to Y \supset Z$; $\to \neg\neg(Y \supset Z)$; where Y and Z are formed using only one $A(x)$, one existential or universal quantifier and zero or more negation symbols.
B. van Rootselaar (Amsterdam)

6705:

Umezawa, Toshio. On an application of intermediate logics. Nagoya Math. J. **16** (1960), 119–133.

Let $P(k, n, p)$ stand for $|a_{n+p} - a_n| < 2^{-k}$ and $\{a_n\}$ be a sequence of rational numbers. Such a sequence $\{a_n\}$ is called a J-, K°- or E-number generator if it satisfies $(\forall k)\ (\exists n)\ (\forall p)\ P(k, n, p)$, $\neg\neg(\forall k)\ (\exists n)\ (\forall p) P(k, n, p)$ or $(\forall k)\neg\neg(\exists n)\ (\forall p) P(k, n, p)$, respectively. In the same way the intuitionistic notions of coincidence, inequality and apartness are weakened. If X, Y, Z stand for any one of J, K°, E, then the species of X-number generators which Y-coincide with a given Z-number generator is called an (X, Y, Z)-real number. Besides the (J, J, J)- and (E, E, E)-real numbers considered in intuitionistic mathematics [cf., e.g., A. Heyting, *Intuitionism*, North-Holland, Amsterdam, 1956; MR **17**, 698] the author finds seven intuitionistically inequivalent notions of real number. Classically all nine notions are equivalent and so they are already in several intermediate logics, e.g., in the author's LE and LD [#6704 above]. For sequences $\{a_n\}$ of rational numbers we have the intuitionistic notion of convergence expressed by

$$(\exists\alpha)(\forall k)(\exists n)(\forall p)(|\alpha - a_{n+p}| < 2^{-k}),$$

α a J-number generator. This notion is weakened both by inserting double negations and by allowing α to range over X-number generators. Thus there are obtained fifteen different notions of Xx-convergence, where X is J, K° or E and $x = 1, \cdots, 5$. The same is done for sequences $\{\alpha_n\}$ of J-number generators instead of sequences of rational numbers. {Reviewer's remarks: A further distinction is obtained by allowing sequences $\{\alpha_n\}$ to range over Y-number generator sequences. If we indicate these notions of convergence by XYx-convergence, then the author's notions are those of XJx-convergence. It must be noted that weakened notions of convergence have been studied in intuitionistic mathematics some time ago, e.g., by J. J. de Iongh [Proc. 10th Internat. Congr. Philos. (Amsterdam, 1948), pp. 744–748; MR **10**, 422] and by J. G. Dijkman [Thesis, Univ. of Amsterdam, 1952; MR **14**, 441]. In particular, Dijkman has some results on JJx-convergence. In def. 17, where the author introduces weakened notions of fundamental sequences, he does not explicitly state for which kind of sequences the notions are intended. Yet it is necessary to do so. If we do this we have JX-, $K^\circ X$-, and EX-fundamental sequences (of X-number generators). XJ-fundamental sequences and their relation

to convergent sequences have been investigated by Dijkman. Finally it should be observed that the proof of theorem 12 (a J-, K°-, and E-fundamental sequence of number-generators are $J1$-, $K^\circ 2$- and $E3$-convergent respectively) is correct only for J-number generators. In particular, statement (A) on p. 132 is incorrect, for if $\{\alpha_n\}$ is an XY-fundamental sequence, where $Y \neq J$, then $\{a^{(n)}\}$ need not be a number generator at all. Correspondingly, theorem 13 has to be restricted to J-number generator sequences. The general situation is even more complicated than appears from the author's valuable results, since besides the stable weakened notions considered by the author other non-stable ones are possible. Moreover, these notions present difficulties of a different kind.}

B. van Rootselaar (Amsterdam)

6706:

Mostowski, A. A class of models for second order arithmetic. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 401–404. (Russian summary, unbound insert)

The author announces several results concerning certain special models of a formal system of analysis developed recently [Grzegorczyk, Mostowski and Ryll-Nardzewski, J. Symb. Logic. **23** (1958), 188–206; MR **21** #4908]. These models are called models absolute for well-orderings and form a narrower class than the ω-models of the previous paper.

A. Rose (Nottingham)

6707:

Shoenfield, J. R. On a restricted ω-rule. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 405–407. (Russian summary, unbound insert)

The author shows that the non-effective ω-rule

$$\text{if } A(0), A(1), \cdots, \text{ then } (x)A(x),$$

which would make formalised arithmetic complete, is equivalent to another rule which is more nearly effective. His rule requires that, given n, there is an effective method of obtaining a proof of $A(n)$ and he gives a precise formulation of this rule, making use of Gödel numbers and recursive functions.

A. Rose (Nottingham)

6708:

Celtin, G. S. Algorithmic operators in constructive complete separable metric spaces. Dokl. Akad. Nauk SSSR **128** (1959), 49–52. (Russian)

Following Šanin the author gives constructive (i.e., recursive) definitions of metric space, of completeness and separability for such spaces and of mapping (algorithmic operator) from one such space to another. His main theorem is a constructive continuity theorem for a constructive mapping ψ of a constructive complete separable metric space A into a constructive metric space B, viz., there exists a recursively enumerable set of pairs $A_{d,n}$, $B_{d,n}$ of spherical neighbourhoods of A, B such that $B_{d,n}$ is of radius 2^{-d} and that if $X \in A_{d,n}$ and $\psi(X)$ is defined then $\psi(X) \in B_{d,n}$; furthermore, for each d and X for which $\psi(X)$ is defined an n can be found such that $X \in A_{d,n}$. Using this theorem he proves that an effective operation mapping the set of general recursive functions into a set of partial recursive functions is (weakly) equal to a partial recursive functional. He quotes Muchnikl as having shown that this is not true for strong equality. (This has been proved independently by Kreisel, Lacombe and Shoenfield [*Constructivity in mathematics* (Colloq., Amsterdam, 1957), pp. 290–297, North-Holland, Amsterdam, 1959; MR **21** #7159; p. 294] who quote Friedberg [C. R. Acad. Sci. Paris **247** (1958), 852–854; MR **20** #6355] for the counter example in the case of strong equality.) He then derives a theorem on the approximability of constructive functions of constructive real numbers by pseudopolygonal functions, from which he draws the corollary that such a function F is continuous in the sense that if X is a constructive real number and $F(X)$ is defined, then for each n an m can be found such that $|F(Y)-F(X)|<2^{-n}$ for all constructive Y for which $|Y-X|<2^{-m}$ and $F(Y)$ is defined. This is a strengthening of theorems of Markov (and Lacombe). The present paper is said to generalise results of the author in Trudy 3 Vsesoyuzn. Mat. S"ezda (Moscow, 1956) Tom 1, pp. 188–189, Izdat. Akad. Nauk SSSR, Moscow, 1956 [see MR **20** #6973a], not available to the reviewer.

J. C. Shepherdson (Bristol)

6709:

Kreisel, G.; Shoenfield, J.; Wang, Hao. Number theoretic concepts and recursive well-orderings. Arch. Math. Logik Grundlagenforsch. **5** (1960), 42–64.

The motivation for this impressive study stems from two well-known methods for relating arithmetic predicates with constructive ordinals. In the method of Markwald [Math. Ann. **127** (1954), 135–149; MR **15**, 771], a central role is played by the set $W(\sigma)$ (as a function of the ordinal σ) of Gödel numbers of recursive well-orderings of ordinals $<\sigma$. Each arithmetic predicate is many-one reducible to some $W(\sigma)$. The method of Kleene [Amer. J. Math. **66** (1944), 41–58; **77** (1955), 405–428; MR **5**, 197; **17**, 5] involves the set $O(\sigma)$ of notations for ordinals $<\sigma$. For each arithmetic predicate A, there is an ordinal σ such that A is uniformly many-one reducible to $O(\tau)$ for every $\tau \geqq \sigma$ (i.e., the same reducing function serves for all τ). The first part of the present paper concerns the relationships among the various sets W and O with respect to Turing reducibility and many-one reducibility.

Results on reducibility are given in terms of degrees (in the sense of Kleene-Post) and many-one degrees (which are defined from the notion of many-one reducibility in the same way as the Kleene-Post degrees are defined from Turing reducibility). Results relating the many-one degrees of $W(\sigma)$'s and $O(\tau)$'s are complete. For each ordinal $\sigma<\omega^\omega$, the degree of $W(\sigma)$ is expressed as the highest degree represented by a predicate in n-quantifier form, while the many-one degree of $W(\sigma)$ is either equated to a many-one degree consisting of the complete predicates in an alternating quantifier form or else bounded by two such many-one degrees. One of the principal theorems gives, for each arithmetic predicate A, a minimal ordinal σ such that A is uniformly many-one reducible to $W(\tau)$ for all $\tau \geqq \sigma$.

In the latter part of the paper, relations between arithmetic propositions and ordinals are expressed in a number-theoretic formal system. This approach culminates in a theorem that every true arithmetic sentence is formally provable by transfinite induction over a recursive well-ordering.

G. F. Rose (Pacific Palisades, Calif.)

6710:

Kreisel, G. Mathematical significance of consistency proofs. J. Symb. Logic **23** (1958), 155–182.

Verf. schlägt in diesem Vortrag vor, das Hilbertsche Programm der (konstruktiven) Widerspruchsfreiheitsbeweise formalisierter Theorien durch ein Programm der "rekursiven Interpretation" nicht-konstruktiver Theorien zu ersetzen.

Die Beispiele, an denen die Methoden—und das bisher Erreichte—des neuen Programms dargestellt werden, sind sehr überzeugend. Es ist leider nur so, daß das neue Programm die Aufgaben des Hilbertschen Programms nicht übernehmen kann, und daß diese Aufgaben dadurch nichts an Wichtigkeit verlieren, obwohl Verf. meint, es sei "hard to imagine, what more we know about ... [consistency], when it has been proved constructively than when it has been proved non-constructively [p. 177]." "Note also, and this seems worth stressing, that we are not rejecting any part of non-constructive work [p. 160]."

Die Formalisierungen nicht-konstruktiver Theorien (ganz unabhängig von ihrer "Berechtigung", nach der ein Mathematiker anscheinend nicht fragen darf) besser zu beherrschen, das ist das einzige Ziel des neuen Programms der rekursiven Interpretation: "this increases our mastery of the subject [p. 160]."

Der "Sinn" dieser unkritischen Formalisierungen bleibt zwar so—auf Grund des Verzichts auf das Hilbertsche Programm—im Dunkeln, aber es lassen sich die Gedankengänge des Verfassers, wenn er nun im Sinne seines Programms vorgeht, doch konstruktiv nachvollziehen: jede Formalisierung kann ja zum Gegenstand der Betrachtung gemacht werden und diese Betrachtung kann dann, bei einiger kombinatorischer Komplikation, auch selbständiges Interesse erhalten.

Der Zusammenhang der Untersuchungen des Verfassers mit der konstruktiven Mathematik ist sogar enger, als es die allgemeine Definition der "rekursiven Interpretation" erwarten läßt. Diese ist die folgende: $\mathfrak{F}$ sei eine Formalisierung der Arithmetik, die aus einer formalisierten Logik $\mathfrak{L}$ und einer Menge von Axiomen besteht, derart daß die Axiome rekursive Formeln sind (d.h. quantorenfreie Formeln, deren nicht-logische Konstanten rekursiv sind). F sei ein Untersystem von $\mathfrak{F}$, das nur rekursive Formeln enthalte. Eine rekursive Interpretation von $\mathfrak{F}$ ordne dann —auf rekursive Weise—jeder Formel $\mathfrak{A}$ von $\mathfrak{F}$ eine Folge $A_1, A_2, \cdots$ von Formeln von F zu und jedem Beweis von $\mathfrak{A}$ in $\mathfrak{F}$ einen Beweis einer Formel A_n in F. Die Formeln A_n sollen dabei der Bedingung genügen, daß aus jedem A_n die Formel $\mathfrak{A}$ mithilfe von $\mathfrak{L}$ allein in $\mathfrak{F}$ beweisbar ist. Dieses allgemeine Schema wird so spezialisiert, daß Formeln der Gestalt $\bigwedge_x \bigvee_y A(x, y)$ durch die Folge der $A(x, \varphi_n(x))$ interpretiert werden sollen, wobei φ_n eine Klasse rekursiver Funktionen durchläuft. Formeln der Gestalt (1) $\bigwedge_x \bigvee_y \bigwedge_z A(x, y, z)$ können nicht entsprechend rekursiv interpretiert werden durch Formeln (2) $\bigwedge_x \bigwedge_z A(x, \varphi(x), z)$, weil (1) klassisch beweisbar sein kann, obwohl die Formel (2) für alle rekursive Funktionen φ falsch ist. Aufgrund der klassischen Äquivalenz $\bigvee_y \bigwedge_z B(y, z) \leftrightarrow \bigwedge_\varphi \bigvee_y B(y, \varphi(y))$ wird daher als rekursive Interpretation von (1) eine Folge $A(x, \Phi_n(x, \varphi), \varphi(\Phi_n(x, \varphi)))$ mit einer Klasse rekursiver Funktionale Φ_n benutzt. Das ist die "no-counterexample-interpretation" des Verf. Die Anwendungen dieser Interpretationsmethode geschehen immer so, daß aus der Ableitung (in $\mathfrak{F}$) von $\bigwedge_x \bigvee_y A(x, y)$ bzw. $\bigwedge_x \bigvee_y \bigwedge_z A(x, y, z)$ eine rekursive Funktion φ_n bzw. ein rekursives Funktional Φ_n konstruiert wird. So ergibt sich z.B. im Fall des Hilbertschen Nullstellensatzes für algebraisch-abgeschlossene Körper eine primitiv-rekursive Funktion als eine obere Schranke für die Grade der gesuchten Polynome. Entsprechendes ergibt sich für die Artinsche Lösung des 17. Hilbertschen Problems für reell-abgeschlossene Körper. Hierbei handelt es sich um Anwendungen, die als Logik den reinen Prädikatenkalkül benutzen. Verf. gibt in einem Anhang für diesen Kalkül eine genaue Darstellung der rekursiven Interpretation. Für die Anwendung auf arithmetische Sätze skizziert Verf. u.a. seine konstruktive Verschärfung einer Littlewoodschen Abschätzung von $\pi(n)$. Die rekursive Interpretation liefert eine Methode, Unabhängigkeitsbeweise zu führen, falls es z.B. gelingt, die Klasse der rekursiven Funktionen φ_n, die bei der Interpretation von Formeln $\bigwedge_x \bigvee_y A(x, y)$ auftreten, so zu charakterisieren, daß für eine bestimmte Formel zwar $\bigwedge_x \bigvee_y \mathfrak{A}(x, y)$ wahr ist, aber $\bigwedge_x \mathfrak{A}(x, \varphi_n(x))$ stets falsch. Es ist dann $\bigwedge_x \bigvee_y A(x, y)$ unableitbar. Verf. geht zum Schluß auf einige Schwierigkeiten ein, die der Anwendung dieser Methode entgegenstehen, skizziert aber auch noch nicht durchgeführte Anwendungen.

P. Lorenzen (Kiel)

6711:

Oberschelp, Arnold. Über die Axiome arithmetischer Klassen mit Abgeschlossenheitsbedingungen. Arch. Math. Logik Grundlagenforsch. **5** (1960), 26–36.

Classes of sentences are characterized that are preserved under passage to subsystems of models, to unions of ascending chains, and to homomorphic images. Classes preserved under formation of direct products were discussed in an earlier paper [same Arch. **4** (1959), 95–123; MR **21** #6330]. For the relation of the present paper to other work (not all of it known to the author at the time of writing) we refer specifically to the review of the earlier paper, and to an expository account by the reviewer [Bull. Amer. Math. Soc. **65** (1959), 287–299; MR **22** #2549]. In the perspective of other work it now seems to the reviewer that the interest of the present paper lies less in the particular results obtained than in the method, which provides a natural and more or less uniform approach to a number of related problems and, as the author observes, uses less heavy machinery than many arguments of a similar nature. For subsystems, typically, with each sentence θ is associated a new sentence $S(\theta)$, containing a new one-place predicate I, which asserts essentially that if θ holds and if the set of elements satisfying I is a closed subsystem, then θ holds also relativized to this subsystem. Thus θ is preserved just in case $S(\theta)$ is a theorem. It follows that the question, undecidable in general, whether arbitrary θ is preserved, becomes decidable under familiar special restrictions on θ. The treatment of chains is parallel, as is that for homomorphic images (where three variants of this concept are considered). In a concluding remark, the author notes the apparent failure of the present method for the study of sentences preserved under the passage to the "complex algebra" of an algebra.

R. C. Lyndon (London)

6712:

Lyndon, R. C. Existential Horn sentences. Proc. Amer. Math. Soc. **10** (1959), 994–998.

Every first order formula is equivalent to a conditional

formula $Q\bigwedge_{i\in I}[\bigwedge_{j\in J_i}\alpha_{ij}\supset\bigvee_{k\in K_i}\beta_{ik}]$, where Q is a string of quantifiers, and the α_{ij} and β_{ik} are atomic formulas indexed by finite sets I, J_i, K_i. Such a formula is called a [strict] Horn sentence if each K_i has at most [exactly] one member. It is shown that every existential sentence that is preserved by the operation of taking direct products of algebras is equivalent to an existential Horn sentence. More precisely, assuming that S and T are existential sentences, the following two statements are proved. (1) If T holds in every direct product of a (possibly empty) set of algebras that satisfy S, then there exists an existential strict Horn sentence U such that S implies U and U implies T. (2) If T holds in every direct product of a nonempty set of algebras that satisfy S, then there exists an existential Horn sentence U such that S implies U and U implies T. *B. Jónsson* (Minneapolis, Minn.)

6713:

Taimanov, A. D. A class of models closed under direct products. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 493–510. (Russian)

Details are presented for results announced earlier [Dokl. Akad. Nauk SSSR **127** (1959), 1173–1175; MR **21** #5554]. A footnote cites two recent papers with which the author became acquainted after his writing; he observes that he has answered certain questions raised by Chang and Morell [J. Symb. Logic **23** (1958), 149–154; MR **21** #3359] and that his examples contradict two theorems announced by Lyndon [#6712 above].

R. A. Good (College Park, Md.)

6714a:

Teodorescu, N. La relation d'égalité dans les théories mathématiques. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 247–254. (Romanian. Russian and French summaries)

6714b:

Teodorescu, N. L'existence de la relation d'égalité et les relations d'équivalence dans une théorie mathématique. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 403–410. (Romanian. Russian and French summaries)

In diesen Aufsätzen wird die Relation der Gleichheit auf die Relationen der Äquivalenz zurückgeführt. Es wird im wesentlichen (Definition 2) die Relation der Gleichheit $\varphi(x, y)$ in einer mathematischen Theorie $\mathcal{T}$ als eine solche Relation der Äquivalenz $A(x, y)$ definiert, die jede Relation der Äquivalenz derselben Theorie impliziert. Doch krankt diese Definition an dem Übelstand, dass es in einer Theorie $\mathcal{T}$ keine Äquivalenzrelation zu geben braucht, die sämtliche Äquivalenzrelationen von $\mathcal{T}$ impliziert. Mit den Worten des Verfassers redend, mögen in einer Theorie $\mathcal{T}$ nur die beiden Äquivalenzrelationen $x\equiv y$ (mod 2) und $x\equiv y$(mod 3) bestehen, und dann besteht keine Gleichheitsrelation in $\mathcal{T}$. Damit ist der Satz 10 des zweiten Aufsatzes hinfällig.

Ferner müssten die Grundrelationen einer Mengentheorie $\in$ (und $\subset$) und $\equiv$ (lies: äquivalent) anstatt $\in$ und $=$ sein, was dem gesunden Menschenverstand widerstrebt. Ferner gilt, wie Verfasser auch zugibt, nicht mehr immer das Gleichheitsaxiom $a=b\rightarrow(A(a)\rightarrow A(b))$ von Hilbert und Bernays [*Grundlagen der Mathematik*, Bd. I, J. W. Edwards, Ann Arbor, Michigan, 1944; MR **6**, 29; S. 165], dass doch einen Grundpfeiler einer jeden egalitären Theorie bildet. Ausserdem kommt hinzu, dass es in den Aufsätzen an einer strengen, in der formalen Sprache der mathematischen Logik formulierbaren Definition mangelt, wann eine Äquivalenzrelation als in einer Theorie $\mathcal{T}$ bestehend zu gelten hat.

Auch ansonsten sind viele Fehler in beiden Aufsätzen vorhanden. So wird bei dem Beweis von Lemma I: $(R(x, y)\ \&\ S(y, z)\Rightarrow S(x, z))\Rightarrow(R(x, y)\Rightarrow S(x, y))$, die Gültigkeit dieses Lemmas künstlich auf die Fälle beschränkt, dass $S(y, z)$ "wahr" ist. Ein ähnlicher Fehler kehrt immer wieder in den Beweisen der beiden Aufsätze. Wir haben durchweg solche Beschränkungen ignoriert. Das Lemma I ist dann auch nicht richtig, wie das Einsetzen von w für $S(x, z)$, w für $R(x, y)$ und f für $S(x, y)$ zeigt.

B. Germansky (Berlin)

6715:

Specker, Ernst. Die Logik nicht gleichzeitig entscheidbarer Aussagen. Dialectica **14** (1960), 239–246.

Ein quantumtheoretisches System von Aussagen ist dadurch gekennzeichnet dass zwei Aussagen im Allgemeinen nicht gleichzeitig entscheidbar (im Sinne von: bewertbar mit Hilfe der üblichen zwei Wahrheitswerten) sind. Es wird hier vorausgesetzt dass Konjunktion und Disjunktion nur für gleichzeitig entscheidbare Aussagen definiert sind.

Es erhebt sich dann die Frage: Kann ein solches System durch Einführung von zusätzlichen—fiktiven—Aussagen so erweitert werden, dass im erweiterter Bereich die klassische Aussagenlogik gilt (wobei für gleichzeitig entscheidbare Aussagen Negation, Konjunktion und Disjunktion ihre Bedeutung beibehalten sollen)? Oder, auf Grund der von Birkhoff und von Neumann angegebenen Isomorphie mit der Gesamtheit der linearen abgeschlossenen Teilräume eines komplexen Hilbertschen Raumes: Ist es möglich, die Gesamtheit der (abgeschlossenen) Teilräume eines Hilbertraumes so in einen Booleschen Verband einzubetten dass die Negation, sowie Konjunktion und Disjunktion soweit sie definiert sind (das heisst für orthogonale Teilräume) ihre Bedeutung beibehalten? Die Antwort auf diese Frage erweist sich als negativ, abgesehen von Ausnahmefällen (die unitären Räume der Dimensionen 1 und 2). *E. W. Beth* (Amsterdam)

6716:

Scott, Dana; Suppes, Patrick. Foundational aspects of theories of measurement. J. Symb. Logic **23** (1958), 113–128.

Im Folgenden seien nur Relative mit endlich vielen definierenden Relationen betrachtet. Eine isomorphabgeschlossene Klasse K von Relativen des Typs s heisst eine Theorie des Messens, wenn es ein numerisches Relativ $\mathfrak{R}$ des Typs s gibt, so dass alle Relative von K in $\mathfrak{R}$ einbettbar sind. In dieser Arbeit werden Fragen der Existenz und Axiomatisierbarkeit von Theorien des Messens behandelt. Man kann leicht durch ein Gegenbeispiel zeigen, dass nicht jede isomorph-abgeschlossene Klasse K von Relativen des Typs s eine Theorie des Messens bildet. In der Arbeit wird geziegt, dass die von Luce eingeführten Semiordnungen mit endlichem Feld

eine Theorie des Messens bilden. Eine Relation R heisst eine Semiordnung, wenn sie folgende Axiome erfüllt:

S1 $\neg xRx$;

S2 $xRy \wedge zRw \rightarrow xRw \vee zRy$;

S3 $xRy \wedge zRx \rightarrow wRy \vee zRw$.

Die zugeordnete numerische Relation wird wie folgt definiert:

$$x \gg y \leftrightarrow x > y+1.$$

Bezüglich der Axiomatisierbarkeit konzentrieren sich die Verfasser auf Theorien des Messens, die durch einen generalisierten Ausdruck axiomatisierbar sind. Man nennt eine Theorie des Messens K endlich, wenn alle Relative in K ein endliches Feld haben. Es gilt der Satz von Vaught: Eine endliche Theorie des Messens K ist durch einen generalisierten Ausdruck axiomatisierbar genau dann, wenn K gegenüber Teilsystembildung abgeschlossen ist, und wenn es eine natürliche Zahl n gibt, so dass für jedes endliche Relativ $\mathfrak{A}$ gilt: Wenn jedes Teilsystem von $\mathfrak{A}$ mit nicht mehr als n Elementen in K liegt, so liegt auch $\mathfrak{A}$ in K. In der Arbeit wird ein Beispiel für eine endliche Theorie des Messens angegeben, die abgeschlossen ist gegenüber Teilsystembildung, aber nicht durch einen generalisierten Ausdruck axiomatisierbar ist. *H. Kiesow* (Oberhausen)

6717:

Tait, W. W. A counterexample to a conjecture of Scott and Suppes. J. Symb. Logic **24** (1959), 15–16.

Scott and the reviewer conjectured [#6716 above] that if S is a sentence in the first-order functional calculus with identity and if S is satisfied by every subsystem of any finite relational system satisfying S, then S is finitely equivalent to a universal sentence, where two sentences are said to be finitely equivalent if they are satisfied by the same finite relational systems. To this conjecture the author constructs as a counterexample a sentence which has two two-place relation symbols, one of which denotes a simple ordering and the other what is essentially the successor relation for this ordering. *P. Suppes* (Stanford, Calif.)

SET THEORY

See also 6716, 6717

6718:

Erdös, P.; Hajnal, A. Some remarks on set theory. VIII. Michigan Math. J. **7** (1960), 187–191.

This note contains further results on independent sets of the reals ($=M$) [cf. Erdös, same J. **2** (1953/54) 51–57, 169–173; Erdös and Fodor, Acta Sci. Math. Szeged **17** (1956), 250–260; **18** (1957), 243–260; MR **16**, 20, 682; **18**, 711; **19**, 1152]. Two typical results are the following. (1) If each picture $S(x)$ is bounded and has outer measure at most 1, then for every positive integer k there exists a set of k independent points. (2) If $2^{\aleph_0}=\aleph_1$ then each graph (with vertices the set M) contains either an infinite complete graph or an independent set of vertices of the second category. {Reviewer's remark: The proof of theorem 3″ has a gap in its 3rd line when it is assumed that there exists a transfinite sequence of distinct points such that $x_\alpha \notin A_\alpha$. As such, theorem 3″ is in doubt.} *S. Ginsburg* (Santa Monica, Calif.)

6719:

Ádám, A. On subsets of set products. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 147–149.

Let $\{A_\lambda | \lambda \in \Lambda\}$ be a family of non-empty, pairwise disjoint sets, the index set Λ having at least two elements. Let G be the Cartesian product of the A_λ, and H a subset of G such that each element of each A_λ occurs in at least one element of H. Five properties about H, too technical to be described here, are stated. Cardinality conditions on Λ are given under which certain of these properties become equivalent. Counterexamples are exhibited to show that the results cannot be sharpened. {Reviewer's note: In property A the author uses the term "classification" instead of the standard term "partition".} *S. Ginsburg* (Santa Monica, Calif.)

6720:

Weaver, Milo W. On the commutativity of a correspondence and a permutation. Pacific J. Math. **10** (1960), 705–711.

The author gives necessary and sufficient conditions for a one-to-one mapping S of a finite set M onto itself to be permutable with a mapping P of M into itself. The mappings P, S are expressible as products of more simple mappings $P=P_1P_2$, $S=S_1S_2$; the above conditions describe the properties of these mappings P_1, P_2, S_1, S_2. M. Novotný [Publ. Fac. Sci. Univ. Masaryk **1953**, 53–64; MR **15**, 782] has solved a more general problem: for a mapping F of an arbitrary set M into itself he has constructed all mappings of M into itself which are permutable with F. The author does not make use of Novotný's results. *O. Borůvka* (Brno)

COMBINATORIAL ANALYSIS

See also 6720, 6737, 7020.

6721:

Schneiderreit, R. Eine mögliche Verallgemeinerung der Fibonaccischen Zahlen. Elem. Math. **15** (1960), 82–84.

Let $f(n)$ denote the number of ways of distributing the numbers $1, 2, \cdots, n$ in two boxes so that not more than two consecutive numbers are in the same box; permuting the numbers in a box or interchanging the boxes does not give a new distribution. Then $f(1)=1$, $f(2)=2$ and $f(n)=f(n-1)+f(n-2)$, so that the $f(n)$ are the Fibonacci numbers. As a generalization the author considers the problem of distributing the numbers $1, 2, \cdots, n$ in m boxes so that not more than m consecutive numbers are in the same box. If $F_m(n)$ denotes the number of distributions of this kind (where again permutations of numbers or boxes is ignored) then $F_m(n)=F_m(n-1)+\cdots+F_m(n-m)$ $(n>m)$. Moreover, $F_m(r)=2^{r-1} (1 \leq r \leq m)$. *L. Carlitz* (Durham, N.C.)

6722:

Ryser, H. J. Compound and induced matrices in combinatorial analysis. Proc. Sympos. Appl. Math., Vol. 10, pp. 149–167. American Mathematical Society, Providence, R.I., 1960.

Ce travail est le développement d'une récente publication de l'auteur [Illinois J. Math. **2** (1958), 240–253; MR **22** #4732], à laquelle on voudra bien se rapporter pour les notations et définitions. [Voir aussi N. G. de Bruijn, Nieuw Arch. Wisk. (3) **4** (1956), 18–35; MR **18**, 183.] Les inégalités appartenant aux fonctions symétriques élémentaires ainsi que les sommes de produits homogènes des racines caractéristiques des matrices hermitiennes non négatives sont étudiées. Le paragraphe 3 contient la généralisation des théorèmes déjà obtenus par l'auteur [loc. cit.] et dont voici l'un des plus importants: Les matrices H et B^* satisfont

$$\operatorname{tr}(C_r(H)) \leq \operatorname{tr}(C_r(B^*)) \quad (1 \leq r \leq v),$$

l'égalité remplaçant l'inégalité pour $r=1$; si $k^*+(v-1)\lambda^* \neq 0$ et si l'égalité a lieu pour quelque r $(1<r\leq v)$, ou si $k^*+(v-1)\lambda^*=0$ et si l'égalité est vérifiée pour r avec $1<r<v$, alors $H=B^*$. La 4[ième] partie est consacrée aux applications: (a) inégalités dans les polynomes de degré ≥ 3; (b) inégalités dans les graphes orientés et leurs matrices des incidences; géométries finies; (c) examen de divers problèmes, résolus ou non, sur les permanents.

A. Sade (Marseille)

6723:

Ryser, H. J. Traces of matrices of zeros and ones. Canad. J. Math. **12** (1960), 463–476.

This paper continues earlier work of the author's [same J. **9** (1957), 371–377; **10** (1958), 57–65; MR **19**, 379, 1153] on combinatorial properties of matrices of zeros and ones. In a class of normalized m by n zero-one matrices we have $r_1 \geq r_2 \geq \cdots \geq r_m$ and $s_1 \geq s_2 \geq \cdots \geq s_n$, where r_i, s_j are respectively row sums and column sums, fixed for the class. The property investigated in this paper is the range of possible values for the trace of such matrices. Explicit values are found. *Marshall Hall, Jr.* (Pasadena, Calif.)

6724:

Paige, L. J.; Tompkins, C. B. The size of the 10×10 orthogonal latin square problem. Proc. Sympos. Appl. Math., Vol. 10, pp. 71–83. American Mathematical Society, Providence, R. I., 1960.

L'introduction contient un bref historique des recherches relatives aux quasigroupes orthogonaux. Cas $n=7$; dénombrement de H. Norton, rectifié par le rev. [Ann. Math. Statist. **22** (1951), 306–307; MR **12**, 665]. Evaluation approximative du temps nécessaire aux machines électroniques pour résoudre ce problème dans le cas du 10[ième] ordre (10^{92} secondes ou $4{,}8\cdot 10^{11}$ heures suivant les cas). La difficulté des dénombrements résidant dans les isomorphismes, les auteurs définissent ces isomorphismes. [La transformation la plus générale préservant l'orthogonalité et comprenant l'isotopie et même l'isomérie comme cas particulier est formulée dans: Sade, Publ. Math. Debrecen **5** (1958), 229–240; MR **20** #5751; Ch. II.] Ensuite la méthode exposée est celle d'Esther Seiden et W. H. Munro et consiste à réduire le nombre des élements du quasigroupe en regardant certains éléments comme identiques, ce qui revient à une espèce d'homomorphisme. Il est clair que, si un système orthogonal cancellable admet une partition régulière dont les blocs soient finis, alors le système quotient est encore orthogonal [op. cit., p. 238]. Enoncé de divers lemmes qui précisent les modalités d'emploi de cette méthode et permettent d'abréger considérablement les essais en éliminant les quasigroupes d'après leur structure (sous-quasigroupes et quasigroupes quotients). *A. Sade* (Marseille)

6725:

Wilansky, Albert. A genesis for binomial identities. Math. Gaz. **43** (1959), 176–177.

The author deduces a number of identities involving binomial coefficients from results such as the following. If g is an even or an odd function defined on $(-1, 1)$ and if $s_n=\int_0^1 t^n g(2t-1)\,dt$, then $\sum_{k=0}^n (-1)^k \binom{n}{k} s_k$ is equal to $+s_n$ or $-s_n$ according as g is even or odd.

L. Mirsky (Sheffield)

6726:

Gregg, C. V. Relations between the sums of powers of the natural numbers. Math. Gaz. **44** (1960), 118–120.

The paper deals with elementary relations between the polynomials $f_0(x), f_1(x), f_2(x), \cdots$ for which $f_r(x)-f_r(x-1) = x^r$ and $f_r(0)=0$. *T. Estermann* (London)

6727:

Cetlin, M. L. Some properties of finite graphs bearing on the transportation problem. Dokl. Akad. Nauk SSSR **129** (1959), 747–750. (Russian)

Let G_n be a finite connected graph with vertices $A_1, \cdots, A_n$. An associated set of integers $q=(q_1, \cdots, q_n)$ is called a loading of G_n. An elementary transport is a transformation $q \rightarrow q'=(q_1, \cdots, q_i-1, \cdots, q_k+1, \cdots, q_n)$, where A_i and A_k are adjacent vertices of G_n. If $p=(p_1, \cdots, p_n)$ is another loading of G_n and if $\sum p_i = \sum q_i$, then q can be transformed into p by a combination of elementary transports. For a given p and q such a combination is called a transport plan along G_n and the number of its elementary transports is called the value of the plan. A plan of minimum value is called a minimal, or optimal, transport plan. In the consideration of transport plans the author confines the treatment to the case $\sum q_i=0$, $p_1=p_2=\cdots=p_n=0$, as representing no loss of generality. A transport plan is determined by the numbers x_{ik} $(i\neq k)$ of elementary transports along edge l_{ik} of G_n $(x_{ik}=-x_{ki})$ and the x_{ik} satisfy $\sum_k x_{ik}=q_i$. For a minimal transport plan the expression $\sum_{i<k} |x_{ik}|$ must attain a minimum.

The author outlines a method of determining minimal transport plans. If G_n is a tree, then by deleting edge l_{ik}, yielding two sub-trees with vertices $A_{i_1}, \cdots, A_{i_{n_1}}$ and $A_{j_1}, \cdots, A_{j_{n_2}}$, with $n_1+n_2=n$, the x_{ik} for a minimal plan is determined by $x_{ik}=\sum_{\alpha=1}^{n_1} q_{i_\alpha} = -\sum_{\alpha=1}^{n_2} q_{j_\alpha}$.

For G_n not a tree, the determination of a minimal transport plan is more complex and rests upon two lemmas, stated but not proved. Lemma 1: The expression $\sum_{j=1}^n |a_j-x|$ for given $a_1, \cdots, a_n$, with $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_n}$, attains a minimum for $a_{i_\alpha} \geq x \geq a_{i_\beta}$, where $\alpha=[\frac{1}{2}(n+1)]$, $\beta=[\frac{1}{2}(n+2)]$. Lemma 2: Consider a cyclic graph with vertices $A_1, \cdots, A_n$, loading $q_1, \cdots, q_n$, and edges $l_{12}, l_{23}, \cdots, l_{n-1,n}, l_{n1}$. Set $x_{12}=q_1-x$, $x_{23}=q_1+q_2-x$, $\cdots$, $x_{n-1,n}=q_1+\cdots+q_{n-1}-x$, $x_{n1}=-x$. Then a minimal transport plan is realized by choosing x in accordance with Lemma 1 with $a_1=q_1$, $a_2=q_1+q_2$, $\cdots$, $a_{n-1}=q_1+\cdots+q_{n-1}$, $a_n=0$.

If the x_{ik} for a given transport plan along a cycle are the ordered numbers $b_1 \geq b_2 \geq \cdots \geq b_n$ and satisfy $b_\alpha \geq 0 \geq b_\beta$,

where $\alpha=[\frac{1}{2}(n+1)]$, $\beta=[\frac{1}{2}(n+2)]$, then the transport plan is called proper. The role of proper transport plans in graphs with cycles is indicated by the theorem: An optimal transport plan along a given graph with cycles induces proper transport plans along each of the cycles of the graph. If the (induced) transport plans along an arbitrary cycle of a graph are proper, then the transport plan along the graph is minimal. (Proof not indicated.)

With the aid of the lemmas and the theorem, an algorithm for construction of a minimal transport plan along a graph containing cycles is outlined as follows. Transports along non-cyclic edges are determined as in the case of trees, reducing the problem to the determination of a minimal transport plan (with altered loading) along a (possibly disconnected) graph, all edges of which are edges in cycles. The transports along these edges are determined by lemma 2, where in some cycles the process is applied repeatedly. The construction of the minimal plan terminates when the transport plans along all cycles of the graph become proper.

The paper concludes with the theorem (proof not indicated): For a given loading q of a graph G_n there exists a tree $D_n \in G_n$ such that a minimal transport plan along G_n coincides with a minimal transport plan along D_n.

R. F. Rinehart (Cleveland, Ohio)

6728:

Wagner, K. Faktorklassen in Graphen. Math. Ann. **141**, 49–67 (1960).

For each vertex v of a given finite graph G let there be defined a non-negative integer $\Gamma(v)$. The author defines a Γ-factor of G as a subgraph in which each vertex x has the valency $\Gamma(v)$. If G has no Γ-factor it is "Γ-prime".

Let us call a pair (p, q) of distinct vertices of G saturated if p and q are joined by at least Min $(\Gamma(p), \Gamma(q))$ distinct edges, and say that the pair (p, p) is saturated if p is incident with at least $[\frac{1}{2}\Gamma(p)]$ loops. The author shows that a Γ-prime graph remains Γ-prime if a new edge is added joining a pair of vertices which is already saturated. He defines a maximal Γ-prime graph as a Γ-prime graph which acquires a Γ-factor whenever a new edge is added joining a pair of vertices not already saturated. He obtains a complete characterization of the maximal Γ-prime graphs, thereby generalizing a result of H.-B. Belck [J. Reine Angew. Math. **188** (1950), 228–252; MR **12**, 730]. From the author's results the known necessary and sufficient conditions for the existence of a Γ-factor can be deduced.

W. T. Tutte (Toronto)

6729:

Kelly, Paul; Merriell, David. A class of graphs. Trans. Amer. Math. Soc. **96** (1960), 488–492.

The graphs considered have no loops and no multiple edges. A graph G of even order is said to have all bisections if for every partitioning of the vertices of G into two equal classes A and B the graphs $G–A$ and $G–B$ are isomorphic. It is proved that the graphs having all bisections are comprised of the following eight types: A complete graph of even order, and the complement of this graph; a graph consisting of two circuits of order 4, and the complement of this graph; a graph consisting of two disjoint equal complete graphs, and the complement of this graph; a graph consisting of two disjoint equal complete graphs and edges so that each vertex of one complete graph is joined to exactly one vertex of the other and conversely, and the complement of this graph.

G. A. Dirac (Hamburg)

6730:

Izbicki, Herbert. Reguläre Graphen beliebigen Grades mit vorgegebenen Eigenschaften. Monatsh. Math. **64** (1960), 15–21.

For a given finite group G let us call a graph X a G-graph if the automorphism group of X is isomorphic to G. It is a conjecture that given any finite G there exist G-graphs X having simultaneously the following three properties: P_k: X has connectivity k; Q_m: X has chromatic number m; R_n: X is regular of degree n; where $k \geqq 1$, $m \geqq 2$, $n \geqq 3$ with the combination $k=1$, $m=2$ excluded. Using some of his earlier results [Monatsh. Math. **61** (1957), 42–50; **63** (1959), 298–301; MR **19**, 161; **21** #4114] the author shows that for any $m \geqq 2$ and $n \geqq 3$ there are infinitely many non-isomorphic G-graphs satisfying both Q_m and R_n. These graphs are connected and for $G \neq 1$ have no fixed vertices or edges.

G. Sabidussi (Hamilton, Ont.)

6731:

Izbicki, Herbert. Graphentransformationen. Monatsh. Math. **64** (1960), 135–175.

The title of this paper is slightly misleading as it does not deal with mappings of graphs but rather with operators on the class of all graphs. Nine such operators ξ are introduced, each assigning to every graph X a unique graph ξX. Among the ξ's are included: the complement (κ), the dual (θ), the removal of loops (τ), the identification of multiple edges (σ), and the subdivision of all edges by a new vertex (η). Considerable space is devoted to the description of some of the properties of each ξX in terms of those of X; in some cases the relations between the automorphism groups of X and ξX are discussed. For graphs without vertices of degree 3 the dual θX is characterized as admitting a natural cover by non-overlapping complete graphs; it is shown that θX uniquely determines X, and that the two graphs have isomorphic automorphism groups. Also, a theorem establishing the existence of graphs with given finite group and certain other properties is proved.

The author uses a representation of graphs by vertices and half-edges rather than the usual vertex-edge notation. He claims that this makes for greater ease in formulating definitions and proving theorems, but even a casual inspection of the paper shows that this is, to say the least, doubtful.

G. Sabidussi (Hamilton, Ont.

6732:

Sabidussi, Gert. Graphs with given infinite group. Monatsh. Math. **64** (1960), 64–67.

It is shown that any group, finite or infinite, can be represented as the automorphism group of a graph.

W. T. Tutte (Toronto)

6733:

Dénes, József. The representation of a permutation as the product of a minimal number of transpositions, and its connection with the theory of graphs. Magyar Tud. Akad. Mat. Kutató Int. Közl. **4** (1959), 63–71. (Hungarian and Russian summaries)

A set T of transpositions in the symmetric group S_n is called regular provided it cannot be written as the union of two sets A and B whose element-sets are disjoint. With each set T of transpositions one can associate a graph G_T whose vertices are the numbers $1, \cdots, n$, two vertices i and j being joined by an edge if and only if the transposition (i, j) belongs to T. The correspondence between T and G_T is one-one. G_T is connected if and only if T is regular. As a consequence of these observations it is proved that the number of representations of a cycle of length n as a product of $n-1$ transpositions is equal to the number of labelled trees with n vertices. The latter number is known to be n^{n-2}. This is used to derive a formula for the number of representations of a given permutation as a product of a minimal number of transpositions.

G. Sabidussi (Hamilton, Ont.)

6734:

Ross, Ian C.; Harary, Frank. The square of a tree. Bell System Tech. J. **39** (1960), 641–647.

Consider a graph G as a reflexive and symmetric relation on its set of vertices and let G^2 be the relational product of G by itself (i.e., G^2 = set of all pairs (a, b) such that (a, c), $(c, b) \in G$ for some c). It is an unsolved problem to characterize those graphs G which have a square root, i.e., for which there is a graph H with $H^2 = G$. The authors solve this problem in the special case where G is the square of a tree, $G = T^2$. T^2 is characterized as admitting a cover by maximal complete subgraphs any two of which intersect in at most one edge and satisfy some further conditions. From this it is proved that if S is a tree with S^2 isomorphic to T^2, then S is isomorphic to T.

G. Sabidussi (Hamilton, Ont.)

6735:

Rényi, Alfréd. Some remarks on the theory of trees. Magyar Tud. Akad. Mat. Kutató Int. Közl. **4** (1959), 73–85. (Hungarian and Russian summaries)

There are many proofs in the literature of the theorem, originally due to Cayley, that the number of trees with n labeled points is (1) $T_n = n^{n-2}$. Let $T_{n,k}$ be the number of forests with $n+k-1$ labeled points $v_1, v_2, \cdots, v_{n+k-1}$ and k connected components (which are trees) such that the first k points are in different components. Cayley generalized (1) by stating without proof (2) $T_{n,k} = k(n+k-1)^{n-2}$. The author proves (2) and obtains still another generalization of (1) by finding an explicit combinatorial formula for the number $G_k(n)$ of labeled forests with n points and k components. In addition, it is shown that the number $T(n, r)$ of labeled trees with n points and r endpoints is $T(n, r) = n!S(n-r, n-2)/r!$, where $S(m, n)$ is the Stirling number of the second kind.

F. Harary (Ann Arbor, Mich.)

6736:

Austin, T. L.; Fagen, R. E.; Penney, W. F.; Riordan, John. The number of components in random linear graphs. Ann. Math. Statist. **30** (1959), 747–754.

The chief purpose of this paper is to find the distribution of the number of connected components in random non-oriented graphs with labelled points and labelled lines; multiple edges are allowed, loops are not. An enumeratory function for the number of graphs with n labelled points, m labelled lines and p connected components, T_{nmp}, is obtained, in which summation over all partitions is involved. In addition, simple expressions are given in some special cases, e.g., $T_{n,\, n-1,\, 1} = (n-1)!n^{n-2}$. Two methods are described for obtaining the average number of components in graphs with n points and m lines, M_{nm}. Convenient approximations for M_{nn} are obtained, e.g., $M_{nn} \doteq 1 + n/e^2$ for $n < e^4$.

G. A. Dirac (Hamburg)

ORDER, LATTICES

See also 6827, 6881, 7106.

6737:

Dilworth, R. P. Some combinatorial problems on partially ordered sets. Proc. Sympos. Appl. Math., Vol. 10, pp. 85–90. American Mathematical Society, Providence, R. I., 1960.

The author has deduced [Ann. of Math. (2) **51** (1950), 161–166; MR **11**, 309] from the following decomposition theorem (1) for partially ordered sets the result (2) of Hall on representatives of sets: (1) The minimal number of chains in the representation of a finite partially ordered set P as a set union of chains is equal to the maximal number n of mutually non-comparable elements of P. (2) There exists a complete system of distinct representatives of the family of sets $T_1, \cdots, T_m$, provided any selection of k of the sets T $(k = 1, \cdots, m)$ contains in their union at least k elements.

Here the author relates (1) in his introductory remarks to theorems of König and of Dantzig and Hoffmann, and the work of Fulkerson. Then he proves (1) using (2) directly, as follows: (a) the proof of (1) is reducible to the case where each element of P belongs to at least one set of n non-comparable elements; (b) in this case, the family of sets of n non-comparable elements forms a distributive lattice L under a suitable ordering relation; (c) If $Q_1, \cdots, Q_m$ is a maximal ascending chain of L, then P is the set union of the Q's. The main theorem is then deduced.

{In discussing the structure of L, the author uses $\vee$, $\wedge$ for set operations and $\cup$, $\cap$ for the lattice operations, contrary to usual practice.}

V. S. Krishnan (Madras)

6738:

Löttgen, U.; Wagner, K. Über eine Verallgemeinerung des Jordanschen Kurvensatzes auf zweifach geordnete Mengen. Math. Ann. **139**, 115–126 (1959).

From the authors' introduction: "Zum ersten Male hat wohl F. Riesz das Problem gestellt, Sätze über Gebietsteilungen in mehrfach geordneten Mengen zu beweisen. In seiner Arbeit Math. Ann. **61** (1905), 406–421, werden Sätze über Approximationenen einer Gebietsgrenze mit Hilfe von Polygonen bewiesen, jedoch unter der einschränkenden Voraussetzung, dass alle Elemente der Gebietsgrenze zugänglich, und zwar von der gleichen Zugänglichkeit sein sollen. Die vorliegende Arbeit enthält keine Voraussetzungen über die Zugänglichkeit der Elemente in der betrachteten zweifach geordneten Menge. Zugelassen werden auch zweifach geordnete Mengen, die (als topologische Räume aufgefasst) nicht metrisierbar sind. Gleichzeitig ergibt sich hierdurch also ein "nicht metrischer" Beweis des Jordanschen Kurvensatzes.

Vorausgesetzt wird lediglich, dass die beiden Koordinatenmengen im Dedekindschen Sinne stetig und ohne Endelemente sind. Diese Voraussetzung ist auch notwendig."

J. Hartmanis (Schenectady, N.Y.)

6739:

Klein-Barmen, Fritz. Zur Theorie der euklidischen Verbände. Math. Ann. **140** (1960), 263–277.

In der Arbeit sind die Resultate der früheren Arbeiten des Autors [J. Reine Angew. Math. **195** (1955), 121–126; MR **18**, 6; und einige andere Arbeiten] vertieft und in einer übersichtlicheren Form dargelegt. Der Verband der natürlichen Zahlen, mit der Teilbarkeitsbeziehung als Ordnungsrelation, wird zum "euklidischen" und "streng euklidischen" Verband verallgemeinert. Es werden Analoga einiger zahlentheoretischen Sätze gefunden, so z.B. des Satzes von der eindeutigen Primzerlegung und des Möbiusschen Umkehrsatzes. Da in einem längenendlichen euklidischen Verband das Intervall $[0, a]$ mit dem Verband der Teiler einer natürlichen Zahl n isomorph ist [siehe MR **18**, 6], sind manche Resultate der Arbeit naheliegend.

M. Kolibiar (Bratislava)

6740:

Maeda, Fumitomo. Decomposition of general lattices into direct summands of types I, II and III. J. Sci. Hiroshima Univ. Ser. A **23**, 151–170 (1959).

The author constructs a general abstract theory of the decomposition of lattices of a certain type into three direct summands. This theory includes a number of decomposition theorems already in the literature as special cases. The lattices L he considers are called Z-lattices; they are complete lattices with center Z which is a complete Boolean sublattice. He assumes for Z the distributive law $x \cap \bigcup_a y_a = \bigcup_a (x \cap y_a)$, which holds in any complete Boolean algebra [G. Birkhoff, *Lattice theory*, Amer. Math. Soc., New York, 1948; MR **10**, 673; p. 165].

A property of elements of L is said to be a P-property if (1) whenever a non-zero element a of L has this property, so does every non-zero element $z \cap a$, with z in the center Z, and (2) the join of any family of elements of L with the property, whose central covers are independent, also has the property. The central cover $e(a)$ of an element a of L is the least element of the center which includes a.

In a Z-lattice with two P-properties P_m and P_f, an element is called finite if it has property P_f, otherwise it is called infinite; it is called minimal if it has property P_m. A lattice is of type I if it has a finite minimal element whose central cover is 1. A lattice is of type II if it has no finite minimal element, but has a finite element whose central cover is 1; and it is of type III if every non-zero element is infinite. The principal theorem of this paper states that every Z-lattice with two P-properties P_m and P_f is the direct sum of sublattices of types I, II, and III.

This theory is applied to the particular cases of complete complemented modular lattices, to complete generalized relatively orthocomplemented lattices, to continuous geometries, and to the lattice of right annihilators of a Baer ring, to obtain decompositions studied by I. Kaplansky, L. H. Loomis, J. von Neumann, and S. Maeda.

O. Frink (Dublin)

6741:

Lorenc, A. [Lorencs, A.] Trivial congruence relations on a lattice. Latvijas Valsts Univ. Zinātn. Raksti **28** (1959), no. 4, 29–31. (Russian. Latvian summary)

Some results are stated on lattices having only the trivial congruence relations. Theorem 5: To every integer $n>2$ there exists a lattice L_n with this property such that L_n has $n+[\frac{1}{2}n]+1$ elements and the maximal length of chains is n. This fails to hold if L_n has fewer elements.

L. Fuchs (Budapest)

6742:

Slowikowski, W. An abstract form of the measure theoretic method of exhaustion. Fund. Math. **48** (1959/60), 79–84.

Es sei X eine teilweise geordnete Menge, $\aleph$ eine Kardinalzahl. Die Menge X heisst '$\aleph$-vollständig' ($\aleph$-complete), wenn jede gerichtete nach oben beschränkte Teilmenge $Z \subset X$ von der Mächtigkeit $\aleph$ in X ein Supremum hat. Die Menge X heisst 'vollständig', wenn sie für jedes $\aleph$ $\aleph$-vollständig ist. Eine vollständige Menge X heisst '$\aleph$-erreichbar' ($\aleph$-accessible), wenn jede nach oben beschränkte gerichtete Teilmenge von X eine gerichtete Teilmenge von der Mächtigkeit $\aleph$ enthält, die dasselbe Supremum hat.

Es seien X, Y teilweise geordnete Mengen, f eine Abbildung der Menge X in die Menge Y. Die Abbildung f heisst 'streng monoton', wenn sie isoton und auf allen Ketten schlicht ist. Sie heisst '$\aleph$-stetig', wenn sie die Suprema aller gerichteten Teilmengen der Menge X von der Mächtigkeit $\aleph$ erhält. Sie heisst 'stetig', wenn sie für jedes $\aleph$ $\aleph$-stetig ist.

Hauptresultat: Es seien X, Y teilweise geordnete Mengen, f eine Abbildung der Menge X in die Menge Y. Wenn X $\aleph$-vollständig, Y vollständig und $\aleph$-erreichbar und f streng monoton und $\aleph$-stetig ist, so ist X vollständig und $\aleph$-erreichbar und f stetig. Die Ergebnisse der Arbeit können zum Beweis des Satzes von Radon-Nikodym benutzt werden.

{Bemerkung des Referenten: In der Definition einer $\aleph$-vollständigen Menge, einer $\aleph$-erreichbaren Menge und einer $\aleph$-stetigen Abbildung sollen gerichtete Teilmengen von der Mächtigkeit $\aleph$ wahrscheinlich durch gerichtete Teilmengen von der Mächtigkeit $\leq \aleph$ ersetzt werden.}

M. Novotný (Brno)

6743:

Onicescu, O. Notes sur les b-algèbres. Rev. Math. Pures Appl. **4** (1959), 345–350.

Es sei A eine Boolesche σ-Algebra (nach Verf. b-Algebra) und m ein nicht negatives, additives und stetiges Mass auf A mit $m(e) < +\infty$, wobei e die Einheit der Algebra A ist. Es bedeute A^a die b-Algebra, die die Atome w_j der Algebra, A, mit $m(w_j) > 0$, erzeugen. Die Anzahl solcher Atome ist bekanntlich höchstens abzählbar. Ein Element $q \in A$ mit $m(q) > 0$ wird vom Verf. als ein Quasiatom bezeichnet, wenn es im folgenden Sinne indefinit zerlegbar ist: Es sei $q = x \cup y$ mit $x \cap y = \phi$ eine beliebige Zerlegung von q, dann soll das eine von den Zerlegungselementen, z.B. das y, das Mass Null und das andere x das Mass von q haben; dabei ist x ein Quasiatom, also auch indefinit zerlegbar. Die Quasiatome lassen sich durch die folgende Regel in höchstens abzählbar vielen Äquivalenzklassen $(q)_j$ einteilen: q' äqu. q'', wenn und nur wenn $m(q' + q'') = 0$, wobei $+$ die symmetrische Differenz bedeutet. Hierbei kann man jeder Äquivalenzklasse $(q)_j$ als Repräsentanten ein Quasiatom q_j zuordnen derart, dass es gilt $q_j \cap q_k = \phi$ für $j \neq k$; für jedes $q \in (q)_j$ existieren ein Quasiatom $q_j^* \leq q_j$ und ein $a \in A$ mit $q = q_j^* \cup a$ und $m(a) = 0$. Die so

erklärten Repräsentanten q_j erzeugen auch eine b-Algebra A^{aq}. Das Element $e^a = \bigcup_i w_i$ bzw. $e^{aq} = \bigcup_j q_j$ ist die Einheit der Algebra A^a bzw. A^{aq}. Ist nun $e^0 = e - (e^a \cup e^{aq}) \neq \phi$, so bildet das System $A^0 = \{x \in A : x \leqq e^0\}$ eine b-Algebra. Da e^a, e^{aq}, e^0 paarweise fremd sind, so ist $A = A^a \cup A^{aq} \cup A^0$ eine direkte Zerlegung in drei Algebren der Algebra A. Ist $e^0 \neq \phi$, so hat die Algebra A^0 die Eigenschaft: für jede reelle Zahl ξ mit $0 < \xi < m(e^0)$ existiert ein $x \in A^0$ mit $m(x) = \xi$.

Es sei B eine b-Algebra; setzt man $S = \{(x, y) \in B \times B : x \cap y \neq \phi\}$ und ordnet man jedem $x \in B$ als Bild die Teilmenge $S_x = \{(z, y) \in S : z \leqq x\}$ zu, so bilden alle S_x, $x \in B$, einen Körper von Teilmengen der Grundmenge S, der zu der b-Algebra B isomorph ist. *D. A. Kappos* (Athens)

6744:

Theodorescu, R. Remarques sur les homomorphismes aléatoires. An. Univ. "C. I. Parhon" București. Ser. Şti. Nat. No. 22 (1959), 55–58. (Romanian. Russian and French summaries)

Unter einem zufälligen Homomorphismus versteht der Verf. eine Schar $\alpha(\cdot, t)$ von σ-Homomorphismen der σ-Algebra $\mathscr{B}$ der Borelschen Teilmengen der Zahlengeraden R in eine σ-Algebra $\mathscr{A}$, wobei t ein Intervall T in R durchläuft. Er macht Gebrauch von dem Sikorski-Loomisschen Isomorphismus ψ einer geeigneten Quotientenalgebra $\mathscr{X}/\mathscr{J}$ auf $\mathscr{A}$, wobei $\mathscr{X}$ eine σ-Mengenalgebra von Teilmengen einer Menge E und $\mathscr{J}$ ein σ-Ideal in $\mathscr{X}$ bildet, und der entsprechenden Darstellung $\alpha(B, t) = \psi(\kappa(\alpha^{-1}(B, t)))$, $B \in \mathscr{B}$, wobei jedes $\alpha(\cdot, t)$ ein $\mathscr{X}$-meßbare Funktion auf E ist und $\kappa(K)$ mit $K \in \mathscr{X}$ die K enthaltende Klasse in $\mathscr{X}/\mathscr{J}$ bedeutet. Unter der Annahme, auf $\mathscr{A}$ liege eine strikt positive Wahrscheinlichkeit vor, wird nun das "Integral der Wahrscheinlichkeit nach" $(p) \int_T a(B, t)dt$ definiert und einige einfache Eigenschaften werden formuliert. *K. Krickeberg* (Heidelberg)

6745:

Wright, Fred B. Polarity and duality. Pacific J. Math. **10** (1960), 723–730.

Let A and B be two Boolean algebras and X, Y the corresponding Boolean spaces. By a polarity of A into B is meant a mapping $\sharp$: $A \to B$ such that (i) $0^\sharp = 1$, and (ii) for any $p, q \in A$, $(p \vee q)^\sharp = p^\sharp \wedge q^\sharp$. Properties of polarities of A and B in connection with corresponding relations of the spaces X and Y are studied [cf. Wright, Portugal. Math. **16** (1957), 109–117; MR **20** #3803]. If the mappings $\sharp$: $A \to B$ and $+$: $B \to A$ define a Galois connection between A and B, then these mappings are polarities. The author shows that a Galois connection is uniquely determined by the polarity $\sharp$, and finds a necessary and sufficient condition for two polarities to define a Galois connection.

G. Birkhoff [*Lattice theory*, Amer. Math. Soc., New York, 1948; MR **10** 673] has described an important example of Galois connection as follows. Let S and T be two sets, and let ρ be a relation from S to T. For any subset $U \subset S$, denote by $U^\sharp$ the set of $t \in T$ such that $u\rho t$ for all $u \in U$. Similarly is defined V^+ for any $V \subset T$. Then the mappings $\sharp$ and $+$ define a Galois connection between the Boolean algebras of all subsets of S and of T. The author shows that each Galois connection between A and B can be obtained by this method; the role of S and T is played by the Boolean spaces X and Y. Conditions are given for a polarity of A into itself to be the complementation mapping $p \to p'$. *M. Kolibiar* (Bratislava)

6746:

Le Blanc, Léon. Dualité pour les égalités booléennes. C. R. Acad. Sci. Paris **250** (1960), 3552–3553.

Author's summary: "Une correspondance biunivoque est établie entre les égalités booléennes sur un ensemble donné et certaines classes de relations d'équivalence sur cet ensemble. La correspondance établie obéit aux lois de la dualité." *C.-C. Chang* (Los Angeles, Calif.)

6747:

Le Blanc, Léon. Les algèbres booléennes topologiques bornées. C. R. Acad. Sci. Paris **250** (1960), 3766–3768.

Author's summary: "Soit B une algèbre booléenne. Une caractérisation est donné pour les topologies sur B qui font de B une algèbre booléenne topologique bornée." *C.-C. Chang* (Los Angeles, Calif.)

6748:

Le Blanc, Léon. Les algèbres de transformation. C. R. Acad. Sci. Paris **250** (1960), 3928–3930.

Author's summary: "Le but principal de cette Note est de montrer que, dans un certain sens, la théorie des algèbres polyadiques peut se rèduire à celle des algèbres de transformation et inversement." *C.-C. Chang* (Los Angeles, Calif.)

6749:

Le Blanc, Léon. Représentation des algèbres polyadiques pour anneau. C. R. Acad. Sci. Paris **250** (1960), 4092–4094.

Author's summary: "Une description purement algébrique est donné des algèbres polyadiques qui correspondent aux théories mathématiques qui sont des extensions de la théorie des anneaux commutatifs avec élément unité et de caractéristique infinie." *C.-C. Chang* (Los Angeles, Calif.)

GENERAL MATHEMATICAL SYSTEMS

6750:

Świerczkowski, S. Algebras independently generated by every n elements. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 501–502. (Russian summary, unbound insert)

The concept of independence referred to in the title is due to Marczewski [same Bull. **6** (1958), 731–736; MR **21** #3363]. An algebra is said to be trivial in case the only operations are the identity operations $f(x_1, \cdots, x_n) = x_k$. The following results are stated without proofs: (I) There is only one non-trivial algebra which is independently generated by every 3 elements. This algebra has 4 elements and all operations are generated by the ternary operation f such that $f(x, y, z) \neq x, y, z$ for x, y, z distinct, and $f(x, x, y) = f(x, y, x) = f(y, x, x) = y$ for all x, y. (II) If an algebra is independently generated by every n elements, where $n \geqq 4$, then it is a trivial algebra with n elements.

(III) Every infinite cardinal is the power of a non-trivial algebra which is independently generated by every 2 elements. A finite cardinal m is the power of such an algebra if and only if m is a power of a prime.

B. Jónsson (Minneapolis, Minn.)

6751:

Marczewski, E. Independence in some abstract algebras. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7 (1959), 611–616. (Russian summary, unbound insert)

In this paper a class of algebras is defined in which the author's concept of independence [see reference in review above] has properties similar to the properties of linear independence in a vector space. Two algebraic operations, f and g, of n variables are said to be distinguishable by the first variable in case there exist elements $a_1, a_2, \cdots, a_n$, a_1' such that $f(a_1, a_2, \cdots, a_n)=g(a_1, a_2, \cdots, a_n)$ and $f(a_1', a_2, \cdots, a_n)\neq g(a_1', a_2, \cdots, a_n)$. A v-algebra is an algebra with the following property: For any algebraic operations f and g in n variables that are distinguishable by the first variable there exists an algebraic operation h in $n-1$ variables such that the equations $f(x_1, x_2, \cdots, x_n)=g(x_1, x_2, \cdots, x_n)$ and $x_1=h(x_2, \cdots, x_n)$ are equivalent. Several elementary properties of v-algebras are proved, in particular a replacement property that insures the existence of a basis and the equality of the cardinals of different bases.

B. Jónsson (Minneapolis, Minn.)

6752:

Urbanik, K. Representation theorem for Marczewski's algebras. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7 (1959), 617–619. (Russian summary, unbound insert)

The algebras referred to in the title are the systems called v-algebras in the paper reviewed above. The following three types of v-algebras $(A, \mathbf{A})$ are considered: (I) A is a vector space over a field K and $\mathbf{A}$ the set of all functions $f: f(x_1, \cdots, x_n)=\sum_{k=1}^{n} \lambda_k x_k + a$ with $\lambda_k \in K$ and $a \in A$. (II) As (I) except that the scalars λ_k are required to satisfy the condition $\lambda_1+\lambda_2+\cdots+\lambda_n=1$. (III) A is a nonempty set, G is a group of transformations of A such that each member of G except the identity has exactly one fixed point, A_0 is a subset of A invariant under G and containing the fixed points, and $\mathbf{A}$ is the set of all functions f defined as follows: $f(x_1, \cdots, x_n)=g(x_j)$ with $1\leq j\leq n$ and $g\in G$, or $f(x_1, \cdots, x_n)=a$ with $a\in A_0$. It is stated without a proof that every v-algebra is isomorphic to an algebra of type I, II or III.

B. Jónsson (Minneapolis, Minn.)

6753:

Goetz, A.; Ryll-Nardzewski, C. On bases of abstract algebras. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 8 (1960), 157–161. (Russian summary, unbound insert)

This paper concerns relations between the number of basis elements in two or more finite bases of an abstract algebra. A basis is an independent set of elements which generate the algebra [see reference in review #6750]. The author shows that the cardinal numbers of all bases of an algebra form an arithmetic progression, and conversely that, given any arithmetic progression of positive integers, there exists an algebra whose bases have just the numbers of this progression for their cardinal numbers. He shows that if two sets of elements of an algebra have a certain relationship of interdefinability, then one set is independent if and only if the other is independent, that both generate the same subalgebra, and that one set is a basis if and only if the other is also. He further shows that if an algebra A has a basis with just one element and also a basis with more than one element, then for every positive integer n there exists in A a minimal set of generators of A consisting of n elements which are self-dependent. An element is self-dependent if the set consisting of just this element is a dependent set.

O. Frink (Dublin)

THEORY OF NUMBERS

See also 6726.

6754:

Sierpiński, W. On representations by a sum of five cubes. Wiadom. Mat. (2) 3, 121–122 (1959). (Polish)

The author proves that every integer can be represented in infinitely many ways by a sum of five cubes of integers.

S. Knapowski (Poznań)

6755:

Sierpiński, W. On certain infinite sequences of natural numbers. Wiadom. Mat. (2) 2, 256–268 (1959). (Polish)

Let, for given positive integers s and n, $f_s(n)$ denote that integer which is obtained by inverting the order of figures in the decimal representation of $n+s$. The author proves a number of results concerning the periodicity of the sequence

$$n, f_s(n), f_sf_s(n), f_sf_sf_s(n), \cdots$$

for some values n, s.

S. Knapowski (Poznań)

6756:

Bialoborski, E. On circular numbers. Wiadom. Mat. (2) 4, 91–92 (1960). (Polish)

Let $K_1=B^{n-1}c_n+B^{n-2}c_{n-1}+\cdots+B^2c_3+B^1c_2+B^0c_1$ be a natural number expressed in a system of notation with the base $B>1$. The numbers $K_2=B^{n-1}c_{n-1}+B^{n-2}c_{n-2}+\cdots+B^2c_2+B^1c_1+B^0c_n$, $K_3=B^{n-1}c_{n-2}+B^{n-2}c_{n-3}+\cdots+B^2c_1+B^1c_n+B^0c_{n-1}$, etc., are called the circular variants of the number K_1. The number K is called circular with respect to a divisor d if all its circular variants are divisible by d. Theorem: If the natural number K has n digits in the system with the base $B>1$, and if d divides the number (K, B^n-1), then K is circular with respect to d. Two corollaries about the existence of circular numbers with respect to d follow.

J. W. Andrushkiw (Newark, N.J.)

6757:

Sierpiński, W. On products consisting only of distinct prime factors. Wiadom Mat. (2) 2, 204–206 (1959). (Polish)

Let products of distinct primes be denoted as Q-numbers. The author proves the following. Theorem 1: There exist arbitrarily long sequences of consecutive positive integers none of which is a Q-number. Theorem 2: There exist infinitely many integers k such that $4k+1$, $4k+2$, $4k+3$ are all Q-numbers.

S. Knapowski (Poznań)

6758:

Rokowska, B. On periodic sequences of natural numbers. Wiadom. Mat. (2) **3**, 41–43 (1959). (Polish)

A sequence of numbers $\{S_k\}$ is said to be periodic if there exist numbers m and t such that $S_k=S_{k+t}$ for $k\geq m$. Let $(c_1, c_2, \cdots, c_n)$ denote the number

$$\alpha = 10^{n-1}c_1+10^{n-2}c_2+\cdots+c_n \quad (0 \leq c_i \leq 9),$$

and let $\bar{a}$ be the number $(c_n, c_{n-1}, \cdots, c_1)$. The author proves if a is an integer and b a power of 2 or 5, then the sequence $\{S_k\}$ defined by $S_1=a$, $S_{n+1}=\bar{S}_n+b$ is periodic.

S. Knapowski (Poznań)

6759:

Satyanarayana, M. Odd perfect numbers. Math. Student **27** (1959), 17–18.

It is still an open question whether or not there exist odd perfect numbers, but there are many results concerning the necessary arithmetical structure of such numbers. In the present paper the author uses a simple and elementary argument to show that an odd perfect number must be of the form $p^{4k+1}N^2$, where p is a prime $\equiv 1 \pmod 4$ and $(N, p)=1$. He then deduces the result of J. Touchard [Scripta Math. **19** (1953), 35–39; MR **14**, 1063] to the effect that an odd perfect number must be $\equiv 1 \pmod{12}$ or $\equiv 9 \pmod{36}$.

L. Mirsky (Sheffield)

6760:

Perisastri, M. A note on odd perfect numbers. Math. Student **26** (1958), 179–181.

The author proves that if n is an odd perfect number, $n=\prod_{r=1}^{k} p_r^{a_r}$, and p_1 is the smallest prime divisor of n, then (a) $1/2<\sum_{r=1}^{k} (1/p_r)<2 \ln (\pi/2)$, and (b) $p_1<(2k/3)+2$.

R. P. Kelisky (Yorktown Heights, N.Y.)

6761:

Gutmann, Hans. Beitrag zur Zahlentheorie verallgemeinerter Primzahlen. Verh. Naturf. Ges. Basel **70** (1959), 167–192.

As a generalization of the sequence of prime numbers, the author studies a sequence of natural numbers, $b_1<b_2<\cdots<b_n<\cdots$, which are prime to one another, and the sets of b-numbers which are free from b-squares, corresponding to the sets of square-free numbers in the usual theory. The author studies the behavior of analogues of classical number-theoretic functions and extends theorems in the analytic theory of numbers. This paper embodies simplifications and improvements of his dissertation [same Verh. **69** (1959), 119–144; MR **21** #2626].

S. Ikehara (Tokyo)

6762:

Lehmer, D. H.; Selberg, S. A sum involving the function of Möbius. Acta Arith. **6** (1960), 111–114.

From the introduction: "Let $\mu(n)$ be the Möbius function. The sum $g(x)=\sum_{n\leq x} \mu(n)/n$ may itself be summed to give $G(x)=\sum_{n\leq x} g(n)$. In this note we show that $G(x)-2$ changes sign infinitely often. Some numerical calculations of the first 56 sign changes are described. These show that these 'zeros' of $G(x)-2$ are remarkably close to being in geometric progression with two exceptions. An heuristic explanation of this phenomenon is given."

A. L. Whiteman (Princeton, N.J.)

6763:

Sierpiński, W. A general formula on integer-valued functions of an integral variable. Wiadom. Mat. (2) **2**, 245–248 (1959). (Polish)

The author proves the following: all integer-valued functions $f(x)$ of an integral variable and only such functions are given by the formula

$$(1) \qquad f(x) = c_0+\sum_{k=1}^{\infty} c_k \binom{x+[\frac{1}{2}k]}{k},$$

where $c_0, c_1, \cdots$ are integers. Furthermore, the representation (1) is unique.

S. Knapowski (Poznań)

6764:

Cohen, Eckford. A class of arithmetical functions in several variables with applications to congruences. Trans. Amer. Math. Soc. **96** (1960), 335–381.

Let r represent a positive integer. In this paper the concepts of even and primitive functions modulo r are extended to functions of several variables. Let $n_1, \cdots, n_k$ denote k integral variables. Then a complex-valued function $f_r=f_r(n_1, \cdots, n_k)$ is defined to be a (relatively) even function of $n_1, \cdots, n_k \pmod r$ provided $f_r=f_r((n_1, r), \cdots, (n_k, r))$ for all n_i $(i=1, \cdots, k)$; f_r is called a primitive function of $n_1, \cdots, n_k \pmod r$ if $f_r=f_r(\gamma(n_1, r), \cdots, \gamma(n_k, r))$ for all n_i, where $\gamma(r)$ denotes the core of r and $\gamma(n, r)=\gamma((n, r))$. (The core $\gamma(r)$ is defined to be the product of the distinct prime divisors of r, $\gamma(1)=1$.)

The author derives Fourier expansions of even and primitive functions f_r in terms of functions involving Ramanujan's sum $c_r(n)$. He then applies these expansions to obtain formulas for the number of solutions of pairs of congruences $\pmod r$ of which at least one is bilinear. With the aid of Cauchy products $\pmod r$ he also obtains similar results for pairs of linear congruences. One of the applications is a simple arithmetical formula for the number of solutions $\theta_r(m, n)$ of the congruences $m\equiv u+x \pmod r$, $n\equiv v+y \pmod r$ such that $(u, v, r)=(x, y, r)=1$. As a corollary it follows that $\theta_r(m, n)>0$ for all values of m, n, r. In conclusion the author establishes analogues in two variables of some of Ramanujan's expansions of arithmetical functions in infinite series.

A. L. Whiteman (Princeton, N.J.)

6765:

Pellegrino, Franco. Sul coniugato mod m di un numero intero. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 318–321.

The author discusses the solvability of the congruence $ax\equiv Q((|a|, m)) \pmod m$, where a, m are integers with $a\neq 0$, $m>0$, and $Q(n)$ denotes the largest square-free integer dividing n.

C. G. Lekkerkerker (Amsterdam)

6766:

Gloden, Albert. Résolution de quelques congruences d'ordre supérieur. Les mathématiques de l'ingénieur, pp. 180–184. Mém. Publ. Soc. Sci. Arts Lett. Hainaut, vol. hors série, 1958.

The congruences considered are

$$(1) \qquad x^4+1 \equiv 0 \pmod p,$$

$$(2) \qquad x^4-x^2+1 \equiv 0 \pmod p,$$

and their solutions are deduced from certain quadratic partitions of the prime p. For example, in the case of (1) suppose that $p=2z^2+t^2=2u^2-v^2$, then the four solutions of (1) are

$$x \equiv \pm z/t \pm u/v \pmod p.$$

For (2), the author supposes that $p=a^2+3b^2$. He then solves the quadratic congruence

$$x^2 \equiv (a+b)/2b \pmod p.$$

Then x, $1/x$, and their negatives are the solutions of (2).
D. H. Lehmer (Berkeley, Calif.)

6767:

Carlitz, L. Some congruences involving sums of binomial coefficients. Duke Math. J. **27** (1960), 77–79.

J. Adem, dans l'étude des relations existant entre générateurs de l'algèbre de Steenrod, a utilisé une congruence concernant les coefficients binomiaux; on voit facilement (dit l'auteur) qu'elle est équivalente à

$$(1)\qquad \sum_{r+s=n} \binom{a+rq}{r}\binom{b+sq}{s} \equiv \binom{a+b+nq}{n} \pmod q,$$

où q est un entier premier, a et b sont entiers. Cette congruence vaut en réalité pour p entier quelconque, a et b étant des fractions dont le dénominateur est premier à q; l'auteur la déduit d'une identité entre séries entières en z:

$$(2)\qquad x^a \equiv \sum_{k=0}^{\infty} \binom{a+kq}{k} z^k \pmod q,$$

où x est la série entière en z définie par $x=1+x^q z$.

L'auteur étudie aussi une autre identité, déduite de (2) lorsque q est une puissance de p premier, z étant remplacé par z^{q-1}; à cette occasion, il donne un développement en série entière de

$$(1+\sum_{k=1}^{\infty} c_k z^{q^k-1})^a$$

pour a entier, $0 \leqq a < p$, et il montre que ce développement n'est plus valable, en général, pour $a \geqq p$.
H. Cartan (Paris)

6768:

Carlitz, L.; Levine, Jack. Some problems concerning Kummer's congruences for the Euler numbers and polynomials. Trans. Amer. Math. Soc. **96** (1960), 23–37.

The Euler polynomial $E_m(a)$ of degree m is the unique polynomial solution of the equation $E_m(a+1)+E_m(a)=2a^m$. Let p be an odd prime, a and c fixed rational numbers which are integral (mod p), and let $c \not\equiv 0 \pmod p$. Put $e_m=c^m E_m(a)$. Then it is well known that the rational numbers e_m satisfy Kummer's congruence

$$(*)\qquad \sum_{s=0}^{r} (-1)^s \binom{r}{s} e_{m+s\mu} \equiv 0 \pmod{(p^m, p^r)},$$

when $\mu=p-1$. In this paper the authors make clear what is meant by saying that (*) is "best possible". Specifically, they answer the following three questions. (i) For $r=1$, what is the least positive integer μ such that $e_{m+\mu} \equiv e_m \pmod p$ for all $m \geqq 1$? (ii) What is the least positive integer μ such that (*) holds for all m and fixed r? (iii) What is the least positive t such that

$$\sum_{s=0}^{t} (-1)^s \binom{t}{s} e_{m+s(p-1)} \equiv 0 \pmod{(p^m, p^r)}$$

for all m and fixed r? They show for (i) and (ii) that $\mu=p-1$; for (iii) they show that $t=r$ provided that $2^p \not\equiv 2 \pmod{p^2}$ and $p \geqq 2r^2+1$. The authors also derive various extensions of these results and discuss a number of related questions. *A. L. Whiteman* (Princeton, N.J.)

6769:

Kirchner, Roger B. The generalized coconut problem. Amer. Math. Monthly **67** (1960), 516–519.

Quel est le plus petit entier A tel que, étant donnés m entiers non négatifs, $V_1, V_2, \cdots, V_m$, les m quotients $U_1=(A-V_1)/n$, $U_2=[(n-1)U_1-V_2]/n$, $\cdots$, $U_m=[(n-1)U_{m-1}-V_m]/n$ soient tous entiers? Expression générale de A; exemples; cas où tous les V ne sont pas positifs ou nuls.
A. Sade (Marseille)

6770:

Leszczyński, B. The equation $n^x+(n+1)^y=(n+2)^z$. Wiadom. Mat. (2) **3**, 37–39 (1959). (Polish)

Theorem: If $y>1$, the equation given in the title of the paper has only two solutions in the natural numbers: (1) $n=1$, x arbitrary, $y=3$, $z=2$; (2) $n=3$, $x=y=z=2$. The author states that the question whether the equation has a solution for $y=1$ remains open (meaning, probably, a solution different from $n=x=y=z=1$).
J. W. Andrushkiw (Newark, N.J.)

6771:

Cassels, J. W. S. On a diophantine equation. Acta Arith. **6** (1960), 47–52.

The theory of the set of rational points on cubics of genus one is used to show that the system of equations $r+s+t=rst=1$ is insoluble in rational numbers.
G. B. Huff (Athens, Ga.)

6772:

Piehler, Joachim. Über Primzahldarstellungen durch binäre quadratische Formen. Math. Ann. **141** (1960), 239–241.

The author proves by elementary methods the following theorems about binary quadratic forms with rational integral coefficients: (1) If two forms with the same discriminant both represent some prime number, then they are properly or improperly equivalent; (2) If p_1 and p_2 are two positive prime integers, then a necessary and sufficient condition for the existence of a positive definite form of discriminant D which represents both of them is that $D=(k^2-4p_1p_2)/h^2$ for some integers $k=0, 1, \cdots, (2p_1p_2)^{1/2}$ and h. Thus there are only finitely many positive definite forms representing both p_1 and p_2.
G. Whaples (Bloomington, Ind.)

6773:

Lehmer, D. H.; Lehmer, Emma. On the cubes of Kloosterman sums. Acta Arith. **6** (1960), 15–22.

Let p be a prime >3 and let $\chi(s)$ be the quadratic character of s modulo p. The p Kloosterman sums

$$S(\lambda, p) = S(\lambda) = \sum_{h=1}^{p-1} \varepsilon^{h+\lambda\bar{h}} \quad (\lambda = 0, 1, \cdots, p-1),$$

where $\varepsilon=\exp(2\pi i/p)$ and $h\bar{h} \equiv 1 \pmod p$ are of two main

types according as $\chi(\lambda)=+1$ or -1. Put $f(\varepsilon)=S(1)$, $g(\varepsilon)=S(N_0)$, where $\chi(N_0)=-1$, and write

$$\sum_{\nu=0}^{p-1}\{f(\varepsilon^\nu)\}^n=\sigma_n,\quad \sum_{\nu=0}^{p-1}\{g(\varepsilon^\nu)\}^n=\sigma_n'.$$

The authors deduce the formulas

$$\sigma_3=\begin{cases}p^2\{2\chi(-1)-1\}+2p & \text{if } p=6n-1,\\ p^2+2p\{1+2\chi(-1)A^2\} & \text{if } p=6n+1=A^2+3B^2,\end{cases}$$

$$\sigma_3'=\begin{cases}-p^2\{1+2\chi(-1)\}+2p & \text{if } p=6n-1,\\ p^2+2p\{1-2\chi(-1)A^2\} & \text{if } p=6n+1=A^2+3B^2,\end{cases}$$

from the identity

$$\sum_{x=1}^{p-1}\sum_{y=1}^{p-1}\chi(x+\bar{y}+1)\chi(\bar{x}+y+1)=$$

$$\begin{cases}2p & \text{if } p=6n-1,\\ 4A^2 & \text{if } p=6n+1=A^2+3B^2.\end{cases}$$

The proof of the identity makes use of an ingenious transformation formula.

A. L. Whiteman (Princeton, N.J.)

6774:

Turán, P. Nachtrag zu meiner Abhandlung "On some approximative Dirichlet polynomials in the theory of zeta-function of Riemann". Acta Math. Acad. Sci. Hungar. **10** (1959), 277–298. (Russian summary, unbound insert)

The author writes

$$U_n(s)=\sum_{\nu\le n}\frac{1}{\nu^s},\quad G_n(s)=\sum_{\nu\le n}\frac{\lambda(\nu)}{\nu^s},\quad H(x)=G_{[x]}\ (1),$$

where $s=\sigma+it$ and $\lambda(\nu)$ is Liouville's function; and he pursues his investigation into the logical relation between the existence of zeros of $\zeta(s)$ and of $U_n(s)$ in portions of the half-planes $\sigma>\frac{1}{2}$ and $\sigma\geqq 1$, respectively, [Danske Vid. Selsk. Mat.-Fys. Medd. **24** (1948), no. 17; MR **10**, 286; quoted as (i)]. Denote by R the Riemann hypothesis (that $\times\zeta(s)$ has no zeros in $\sigma>\frac{1}{2}$) and by $\{\delta_n; a_n, b_n\}$ the statement that $U_n(s)$ has no zeros in the half-strip $\sigma\geqq 1+\delta_n$, $a_n\leqq t\leqq b_n$. Let $c_1, c_2, \cdots$ denote positive numerical constants, and $c_1(\cdots), \cdots$ positive numbers depending only on the parameters shown. The main results take the forms: (I) $\{\delta_n; \gamma_n, \gamma_n+T_n\}$ (some $\gamma_n\geqq 0$; all $n>c_1$) $\Rightarrow R$; (II) $R\Rightarrow\{0; c_2, T_n'\}$ (all $n>c_1$); with explicit δ_n, T_n, T_n'. The author's values are

$$\delta_n=n^{-1/2}\log^3 n,$$
$$T_n=e_1(n^3),$$
$$T_n'=e_2(c_3\surd(\log n\log\log n)),$$

where $e_1(x)=e^x$, $e_2(x)=e_1(e^x)$; but he indicates possibilities of improvement. He also states (sometimes with a sketch of proof) analogous results for other Dirichlet polynomials in place of $U_n(s)$, such as its first Çesàro mean.

As in (i), the proof of (I) is in two stages. By a combination of Kronecker's theorem on diophantine approximation and Rouché's theorem on zeros of analytic functions (after the manner of H. Bohr) a connection is first established between zeros of $U_n(s)$ and $G_n(s)$; from this and the hypothesis of (I) it is then deduced that $H(x)\geqq -x^{-1/2}+\varepsilon$ (any $\varepsilon>0$; $x>c_1(\varepsilon)$), and the conclusion that $\zeta(2s)/\{(s-1)\zeta(s)\}$ is regular for $\sigma>\frac{1}{2}+\varepsilon$ follows by an application of Landau's theorem on singularities of functions represented by Dirichlet integrals. [In (9.4) the first term on the right-hand side should have a factor $(s-1)$ in the denominator; the corresponding formula (14.1) of (i) is correct.] The main novelty in the proof is the use of a localized form of Kronecker's theorem when the linearly independent numbers occurring in it are logarithms of distinct primes; this is based on an idea of Bohr and Landau. The author notes, as in (i), that the conclusion of (I) still holds even if the hypothesis fails for an infinity of n's, provided that the number of exceptional $n\leqq x$ is $o(\log x)$ as $x\to\infty$; this comes by an application of a theorem of Pólya in place of Landau's theorem.

Implications in the direction (II) are new. The proof is by two applications of contour integration. In the first, an estimate of $\log\zeta(s)$ and thus of $\zeta(s)$ is deduced from the hypothesis R by the use of a formula for $\zeta'(s)/\zeta(s)$ due to Littlewood. In the second, this estimate is applied to the study of $U_n(s)$ regarded as a coefficient sum in the expansion of $\zeta(s+z)$ as a Dirichlet series in z.

A. E. Ingham (Cambridge, England)

6775:

Newman, D. J. Estimate of a certain least common multiple. Michigan Math. J. **7** (1960), 75–78.

This paper discusses the problem of finding an estimate for the maximum possible value $\Phi(N)$ of the l.c.m. of a set of numbers $N_i\leqq N$, not necessarily unequal, which satisfy $\sum 1/N_i=1$. It is proved, using the prime number theorem, that $\log\Phi(N)\sim N/\log N$.

C. B. Haselgrove (Manchester)

6776:

Sekanina, Milan. Notes on the factorisation of infinite cyclic groups. Czechoslovak Math. J. **9** (84) (1959), 485–495. (Russian. English summary)

The infinite cyclic group in question is taken to be the additive group $\mathfrak{L}$ of integers. If M, A and B are subsets of $\mathfrak{L}$, the equation $M=A+B$ is defined to mean that every m can be expressed uniquely as $m=a+b$ where $a\in A$, $b\in B$. The sets A, B are then called factors of M. If $A+\{a\}=A$ for some $a\neq 0$, A is called periodic. The main result proved is the following. Let $M=\{a_n\}$ be an increasing sequence of integers with $a_1=1$ and with the property that there exist an integer N and a real number $\alpha>2$ such that, for all $n\geqq N$, we have $a_{n+1}-a_n\geqq a_n-a_{n-1}\geqq\alpha^n-\alpha^{n-1}$. Let Z be any finite (possibly empty) subset of $\mathfrak{L}$. Then M is a factor of $\mathfrak{L}-Z$. Further, all the factors of the set of non-negative integers are determined, using Cantor's elementary number system in which a positive integer is expressed in the form $x=\sum_{i=1}^n h_ik_i$, where $0\leqq h_i<m_i=k_{i+1}/k_i$ (m_i a positive integer, $k_1=1$).

R. A. Rankin (Glasgow)

6777:

Supnick, Fred; Cohen, H. J.; Keston, J. F. On the powers of a real number reduced modulo one. Trans. Amer. Math. Soc. **94** (1960), 244–257.

Let $\alpha>1$ and put $v_k=\alpha^k-[\alpha^k]$ ($k=1, 2, \cdots$). In this paper, the distribution of the v_k is investigated, more precisely the decomposition of the set of positive integers into classes C_ν ($\nu=1, 2, \cdots$) such that j, k belong to the same class if and only if $v_j=v_k$. A class C_ν is called unitary or binary if it consists of one or two integers respectively. It is easily shown that there can be nonunitary classes only if α is an irrational algebraic integer whose minimal

polynomial $M_a(x)$ has a negative constant term. Let $L(\alpha)$ be the number of nonzero terms of $M_a(x)$. The authors now derive seven theorems. Some of them give sufficient conditions in order that all classes be unitary. Further, there is a complete discussion of the cases $L(\alpha)=2, 3$. It is also shown that there can be only finitely many binary classes and that in the case $L(\alpha)\geqq 3$ each class is unitary or binary. A set of α's is given for which $L(\alpha)>3$ and not all classes are unitary. *C. G. Lekkerkerker* (Amsterdam)

6778:

Newman, Morris. Periodicity modulo m and divisibility properties of the partition function. Trans. Amer. Math. Soc. **97** (1960), 225–236.

The author conjectures that for every pair of natural numbers m, r, there are infinitely many natural numbers n such that $p(n)\equiv r \pmod m$, where $p(n)$ is the number of unrestricted partitions of n. He proves particular cases of this conjecture, e.g., all those in which $m=5$ or 13. He also obtains similar results involving arithmetical functions other than $p(n)$. *T. Estermann* (London)

6779:

Sierpiński, W. On a certain consequence of the Goldbach hypothesis. Wiadom. Mat. (2) **3**, 22–23 (1959). (Polish)

The author deduces from the Goldbach hypothesis (that every even integer >2 is a sum of two primes), and without referring to Dirichlet's theorem on arithmetical progressions, the following: For every even integer $2k>0$ and every integer $m>0$, there exist arbitrarily large primes p and q such that $2k\equiv p+q \pmod m$. The author remarks that this theorem has been proved, without hypotheses but using Dirichlet's theorem, by A. Schinzel [Compositio Math. **14** (1959), 74–76; MR **21** #2633]. *S. Knapowski* (Poznań)

6780:

Ankeny, N. C.; Chowla, S. A note on the class number of real quadratic fields. Acta Arith. **6** (1960), 145–147.

Let $h=h(p)$ denote the class number of the real quadratic field $R(\sqrt{p})$, where p is a prime $\equiv 1 \pmod 4$. It is proved using the formula

$$\varepsilon^{2h} = \prod_b \sin(b\pi/p)/\prod_a \sin(a\pi/p)$$

where $(a/p)=1$, $(b/p)=-1$, that $h<p$. It is also noted that as a consequence of the 'extended Riemann hypothesis' it follows that

$$h = O(\sqrt{p}\cdot\log\log p/\log p).$$

L. Carlitz (Durham, N.C.)

6781:

Honda, Taira. On absolute class fields of certain algebraic number fields. J. Reine Angew. Math. **203** (1960), 80–89.

From Introduction: "Let P be an algebraic number field of finite degree and K a cyclic extension of prime degree l over P such that the number of ambiguous ideal classes is equal to 1. Moreover, assume that the absolute class field L of K is cyclic over K. Our main theorem asserts that the absolute ideal class group K_L of L is isomorphic to the l-fold direct product of the absolute class group K_Ω of a certain subfield Ω of L, if K/P is ramified, and to the l-fold direct product of a subgroup of index l of K_Ω, if K/P is unramified."

The author proves this using class field theory, some special properties of the Galois group of L/P in this situation, and an interesting lemma of elementary number theory. The field Ω is the fixed field of the unique cyclic subgroup of order l of the Galois group of L/P.

Further result (Theorem 5): Let K be an imaginary quadratic field with a prime discriminant, and assume that the absolute class field L of K is cyclic over K. Then, if $\{\varepsilon_1, \cdots, \varepsilon_m\}$ is a system of fundamental units of the maximally real subfield of L, $\{\varepsilon_1, \cdots, \varepsilon_m, \varepsilon_1^\sigma, \cdots, \varepsilon_m^\sigma\}$ forms a system of fundamental units of L. Here σ is the generating substitution of L/K.

This is proved using results of Ishida [Proc. Japan. Acad. **33** (1957), 293–297; MR **19**, 943] and Kuroda [Nagoya Math. J. **1** (1950), 1–10; MR **12**, 593]. The paper concludes with some numerical examples. *G. Whaples* (Bloomington, Ind.)

6782:

Perron, Oskar. ★Irrationalzahlen. 4te, durchgesehene und ergänzte Aufl. Göschens Lehrbücherei, I. Gruppe, Bd. I. Walter de Gruyter & Co., Berlin, 1960. viii+204 pp. DM 28.00.

The changes in this new edition of this well-known book are mainly confined to the first three chapters, where Dedekind's theory for the foundation of real numbers and related topics are treated. The general plan is unchanged, but there are several improvements which make this part still more readable. The remainder of the book, however, shows only minor alterations and additions.

The fourth chapter contains not only an excellent introduction to the theory of continued fractions, but also the representation of real numbers by the series of Cantor, Lüroth, Engél and Sylvester. As such it still stands apart from all other elementary textbooks I know.

The remaining two chapters contain short introductions to diophantine approximations and to transcendental numbers. The treatment of these subjects is masterly, but the reviewer deplores that these chapters—in view of important recent additions to these theories—have not been worked out somewhat more. For example, there is nothing in the present book about geometry of numbers. Nonetheless, the book is an excellent introduction for a student to these fields in number theory.

{Correction: The statement on p. 194 that we do not know if π is a Liouville number, is no longer true. Mahler proved in 1953 that $|\pi-p/q|>q^{-42}$ for any pair of integers p, q with $q\geqq 2$ [Nederl. Akad. Wetensch. Proc. Ser. A **56** (1953), 30–42; MR **14**, 957].} *J. Popken* (Amsterdam)

6783:

Sudan, Gabriel. ★Geometrizarea fracţiilor continue [Geometrization of continued fractions]. Editura Tehnică, Bucharest, 1959. 436 pp. Lei 17.00.

This interesting text, in Roumanian, deals with the general theory of numerical continued fractions and their use in approximations by rational numbers. Here the proofs of many of the theorems on such continued fraction expansions are carried out by means of geometrical representations.

In chapters 1 and 2 are given the basic theory and

theorems on regular numerical continued fractions, on numerical approximations of real numbers by rational numbers, and considerations of diophantine equations. The discussion in these two chapters parallels the work of O. Perron [*Die Lehre von den Kettenbrüchen*, Bd. I, 3te Aufl., Teubner, Stuttgart, 1954; MR **16**, 239; chapters 2 and 3]. In chapter 3 many properties of regular continued fractions given in the preceding chapters are interpreted and demonstrated geometrically. Chapter 4 is concerned with the extension to two dimensions and the problem of simultaneous approximations of irrational numbers by rational fractions. Chapter 5 deals with the representation of the numerators and denominators of the approximants in two-dimensional space. This is used as a means for the proof of a number of theorems in chapters 1 and 2, in particular, on diophantine equations. Chapter 6 gives more applications and a geometrical proof of a theorem on the numerical approximation of an irrational by a rational number. In chapter 7, extensions to general numerical continued fractions and semi-regular continued fractions are considered. A number of recurrence formulas and limit theorems are proved geometrically. Chapters 8, 9, and 10 deal with geometrical proofs and interpretations of more theorems on number theory and numerical continued fractions. There is a bibliography of 123 references to material given in the book. *E. Frank* (Chicago, Ill.)

FIELDS

6784:

Burnside, William Snow; Panton, Arthur William. ★The theory of equations: With an introduction to the theory of binary algebraic forms. 2 volumes. Dover Publications, Inc., New York, 1960. xiv + 286 + x + 318 pp. $1.85 per vol.

A reproduction of the seventh edition [vol. 1, Longmans, Green, London 1912; vol. 2, Longmans, Green, London 1928].

6785:

Segre, Beniamino. Sulla teoria delle equazioni e delle congruenze algebriche. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **27** (1959), 155–161, 303–311.

In the first note, the number of solutions of an equation $F(x)=0$ which are nth roots of unity (n not a multiple of the characteristic) is expressed as the rank of a "circulant" matrix; this is transformed in various ways. The second note treats the case of a finite field F_q with q elements; by means of the elementary relation $\sum x^i = 0$ or -1 (sum extended to all elements of F_q) according as i is not or is a multiple of $q-1$, it is shown that the number modulo p (p = characteristic) of solutions of a system of equations $P_h(x_1, \cdots, x_n)=0$ in F_q can be expressed as a polynomial in the coefficients of the equations. Some special cases are discussed briefly. *A. Weil* (Princeton, N.J.)

6786:

Segre, Beniamino. Sul numero delle soluzioni di un qualsiasi sistema di equazioni algebriche sopra un campo finito. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 271–277.

This note gives an elementary result on the number of solutions of an overdetermined system of a special type over an arbitrary field, in terms of the rank of a certain matrix. In spite of the title, the note gives no significant application to equations or systems over finite fields. *A. Weil* (Princeton, N.J.)

6787:

Lugowski, Herbert; Weinert, Hanns Joachim. ★Grundzüge der Algebra. Teil III: Auflösungstheorie algebraischer Gleichungen. Mathematisch-Naturwissenshaftliche Bibliothek, 11. B. G. Teubner Verlagsgesellschaft, Leipzig, 1960. 274 pp. (1 insert) DM 13.80.

For Teil I (1957) and Teil II (1958) see respectively MR **19**, 728 and MR **20** #5770. This last volume contains the following chapters: IX Polynome, X Algebraische Körperweiterungen, XI Galois-Theorie; plus many exercises with solutions.

6788:

Dehn, Edgar. ★Algebraic equations: An introduction to the theories of Lagrange and Galois. Dover Publications, Inc., New York, 1960. xi + 208 pp. $1.45.

A corrected republication of the first edition [Columbia Univ. Press, New York, 1930]; emphasis is on Lagrange and Galois, with mention of Abel and no mention of Ruffini. *I. Barsotti* (Providence, R.I.)

6789:

Faddeev, D. K.; Šmidt, R. A. Conditions for imbeddability of a field in the case of a cyclic normal subgroup of the eighth order. Vestnik Leningrad. Univ. **14** (1959), no. 19, 36–42. (Russian. English summary)

Es sei k eine normale algebraische Erweiterung eines Körpers k_0 und F die zugehörige Galois-Gruppe. Gegeben sei ferner eine Gruppe G und eine homomorphe Abbildung von G auf F, deren Kern die zyklische Gruppe von der Ordnung acht ist. Es wird die Frage gelöst, unter welchen Bedingungen kann der Körper k eingebettet werden in einen Körper K, so daß G die Galois-Gruppe von K über k_0 ist und das natürliche Homorphismus der Gruppe von K über k_0 auf die Gruppe von k über k_0 mit dem gegebenen Homomorphismus von G auf F übereinstimmt. Es sind notwendige und hinreichende Bedingungen für eine derartige Einbettung gefunden im Fall, wenn k_0 eine von 2 verschiedene Charakteristik besitzt. *M. Kolibiar* (Bratislava)

6790:

★Séminaire Krasner: année 1953/54. Séminaire de la théorie des corps values. Faculté des Sciences de Paris, Paris, 1954.

The articles 6 to 19 of this seminar are reviewed separately (see the following 14 reviews). The first five articles were reviewed as MR **20** #3858, 3859.

6791:

Aubert, K. E. Corps valués de Krull. Séminaire Krasner 1953/54, Exp. 6, 20 pp. Fac. Sci. Paris, Paris, 1954.

Discussion of the basic properties of valuations with general value groups, with four appendices by M. Krasner on the existence of value groups of prescribed (ordinal) rank, the definition of ideals, problems of notation and the theory of skeletons. [See the earlier exposé no. 5; MR **20** #3859.] *O. F. G. Schilling* (Chicago, Ill.)

6792:

Fleischer, I. Caractérisation topologique des corps valués. Séminaire Krasner 1953/54, Exp. 7, 11 pp. Fac. Sci. Paris, Paris, 1954.

Topological characterizations of valuations and pseudo-valuations based essentially on the work of Kaplansky, Zelinsky and the author. [See Kaplansky, Duke Math. J. **9** (1942), 303–321; MR **3**, 264; Zelinsky, Bull. Amer. Math. Soc. **54** (1948), 1145–1150; MR **10**, 426; and the author, C. R. Acad. Sci. Paris **236** (1953), 1320–1322; MR **14**, 720.] *O. F. G. Schilling* (Chicago, Ill.)

6793:

Fleischer, I. Corps maximalement complets. Séminaire Krasner 1953/54, Exp. 8, 10 pp. Fac. Sci. Paris, Paris 1954.

Discussion of maximally complete fields by means of pseudo-convergent sequences with some simplifying remarks by M. Krasner. [See I. Kaplansky, Duke Math. J. **9** (1942), 303–321; MR **3**, 264.]

O. F. G. Schilling (Chicago, Ill.)

6794:

Lazard, M. Détermination des anneaux p-adiques et π-adiques dont les anneaux des restes sont parfaits. Séminaire Krasner 1953/54, Exp. 9, 16 pp. Fac. Sci. Paris, Paris, 1954.

Discussion of unramified p-adic rings (existence and uniqueness) with emphasis of Teichmüller's systems of multiplicative representatives and Witt's vector calculus, perfect residue class fields being assumed.

O. F. G. Schilling (Chicago, Ill.)

6795:

Lazard, M. Construction des corps valués complets hétérotypiques non ramifiés à partir de leurs corps des restes (non supposés parfaits). Séminaire Krasner 1953/54, Exp. 11, 10 pp. Fac. Sci. Paris, Paris, 1954.

Existence of unramified p-adic fields with imperfect residue class fields according to Teichmüller and Witt.

O. F. G. Schilling (Chicago, Ill.)

6796:

Hertzig, D. La méthode de MacLane pour l'étude des extensions non ramifiées des corps (discrètement) valués complets. Séminaire Krasner 1953/54, Exp. 12, 13 pp. Fac. Sci. Paris, Paris, 1954.

MacLane's theory of unramified complete fields, i.e., emphasis on Hensel's Lemma for the existence and uniqueness proofs. *O. F. G. Schilling* (Chicago, Ill.)

6797:

Samuel, P. Généralités sur l'algèbre locale. Séminaire Krasner 1953/54, Exp. 13, 5 pp. Fac. Sci. Paris, Paris, 1954.

Discussion of valuation rings (rank one and discrete) from the viewpoint of the theory of local (and graded) rings. Concept of the complete tensor product of local rings. *O. F. G. Schilling* (Chicago, Ill.)

6798:

Krasner, M. Prolongement des valuations. Introduction: idée des différentes méthodes et leur comparaison. Séminaire Krasner 1953/54, Exp. 14, 6 pp. Fac. Sci. Paris, Paris, 1954.

A henselian (relatively complete) field is defined as one whose valuation has a unique prolongation to any algebraic extension. Discussion of the existence of prolongations, characterization of henselian fields as fields whose irreducible polynomials have Newton polygons which are straight line segments. Use of pseudo-convergent sequences for the construction of prolongations in transcendental extensions of the base field.

O. F. G. Schilling (Chicago, Ill.)

6799:

Guérindon, J. Espaces vectoriels normés sur un corps valué complet, et unicité du prolongement de sa valuation dans ses extensions algébriques de degré fini. Séminaire Krasner 1953/54, Exp. 15, 3 pp. Fac. Sci. Paris, Paris, 1954.

Use is made of normed vector spaces over a field with a valuation in order to show the uniqueness of the prolongation of a valuation of a complete (rank one) field to a finite algebraic extension.

O. F. G. Schilling (Chicago, Ill.)

6800:

Fleischer, Isidore. Espaces vectoriels sur un corps valué de Krull. Séminaire Krasner 1953/54. Exp. 15 bis, 3 pp. Fac. Sci. Paris, Paris, 1954.

Discussion of Nachbin's theory of the topology of finite dimensional vector spaces over a topological space in relation to the theory of prolongations of complete fields. [L. Nachbin, Bull. Amer. Math. Soc. **55** (1949), 1128–1136; MR **11**, 368.] *O. F. G. Schilling* (Chicago, Ill.)

6801:

Krasner, M. Unicité de prolongement des valuations dans les extensions algébriques des corps valués métacomplets de Krull. Séminaire Krasner 1953/54, Exp. 15 ter, 9 pp. Fac. Sci. Paris, Paris, 1954.

Generalization of the uniqueness theorem for the prolongation to a finite extension of a complete field with a valuation by emphasizing the topology of the base field which is determined by the ideals in the valuation ring.

O. F. G. Schilling (Chicago, Ill.)

6802:

Fleischer, I. Prolongement des valuations. Séminaire Krasner 1953/54, Exp. 16, 2 pp. Fac. Sci. Paris, Paris, 1954.

Existence of a prolongation to an extension field of a field with a valuation is proved by means of Zorn's Lemma; that is, a maximal element in the set of rings in the extension, whose intersection with the base field equals the valuation ring of the given valuation, determines a prolongation. *O. F. G. Schilling* (Chicago, Ill.)

6803:

Hertzig, D.; Krasner, M.; Fleisher, J. [Fleischer, I.] Lemme de Hensel et ses généralisations; lemme de Hensel dans les anneaux locaux. Séminaire Krasner 1953/54, Exp. 18–18A, 4 pp. Fac. Sci. Paris, Paris, 1954.

Proof of Hensel's lemma and some of its elementary consequences for complete local rings.

O. F. G. Schilling (Chicago, Ill.)

6804:

Fleischer, Isidore. Valuations sur les corps premiers. Séminaire Krasner 1953/54, Exp. 19, 2 pp. Fac. Sci. Paris, Paris, 1954.

Proof that prime fields have none but the classical valuations, i.e., for finite characteristic only the trivial valuation, for the field of rational numbers the p-adic valuations. *O. F. G. Schilling* (Chicago, Ill.)

6805:

Ribenboim, Paulo. On the theory of Krull valuations. Bol. Soc. Mat. São Paulo **11** (1959), i–vi, 1–106. (Portuguese)

In this thesis the author considers extensions of Krull's general theory of valuations [J. Reine Angew. Math. **167** (1932), 160–196]. Two main questions are tackled. (I) How are the decomposition, inertial and ramification fields (the latter with degree relatively prime to the characteristic of the residue class field) of a prolongation $\tilde{w}$ in a finite algebraic extension $\tilde{K}$ of a field K with valuation w related to the similar fields corresponding to valuations of $\tilde{K}$ less fine than $\tilde{w}$ (their valuation rings containing that of $\tilde{w}$), and to valuations induced in the various residue class fields belonging to such less fine valuations? (II) Extensions of the approximation theorems for finite sets of valuations (Chinese Remainder Theorem type problems), where the valuation rings have common nonzero prime ideals. As to (II) simple necessary and sufficient conditions for compatibility of the orders of approximation are found [see, for example, theorem 12, p. 216]. The results of (II) are necessary for the consideration of the various decomposition groups associated to $\tilde{w}$ and its less fine valuations. It is noted that the distinct valuations over w are always conjugate though the classical theorem on their number in relation to the number of distinct imbeddings of $\tilde{K}$ into the algebraic closure of a suitable "completion" of K need not hold [see lemma 44, p. 75]. Sufficient conditions for the ramification formula $[\tilde{K}:K]=$ "$\sum_i e_i f_i$" are given [see theorem 16, p. 79]. Question (I) emphasizes various types of henselian (relatively complete) fields, that is, fields K whose valuation has a unique prolongation to $\tilde{K}$. Maximally complete fields and complete fields are characterized by the existence of pseudo-limits of pseudo-convergent and distinguished pseudo-convergent sequences (breadth is a prime ideal). Sample results are (i) if $\tilde{K}$ is an immediate maximal completion of K with respect to W, then $\tilde{K}$ contains an immediate maximal completion for every valuation which is less fine than w, and (ii) maximal completeness of K implies maximal completeness of the residue class field of a less fine valuation [see lemmas 9, 10, pp. 9, 10]. In lemma 25 (p. 29) it is shown that the extension $\tilde{K}$ is complete if K is complete and if furthermore (a sufficient condition) the valuation w has no proper limit prime ideal (e.g., if w has finite rank). The methods of proof are elaborations of those of Krull, Ostrowski and Kaplansky.

O. F. G. Schilling (Chicago, Ill.)

ABSTRACT ALGEBRAIC GEOMETRY

See also 6771, 6851, 7008, 7029, 7149.

6806:

Boughon, P. Propriétés différentielles des variétés algébriques définies sur un corps de caractéristique $p>0$. Ann. Fac. Sci. Univ. Toulouse (4) **21** (1957), 185–253 (1959).

Chapter I is devoted to a study of the local parametrisation (in formal power series, thought of as power series) of an algebraic variety over a field of characteristic $p>0$, at its generic point, and of the possible gaps in the series. The principal geometrical result is that the order of contact of the tangent space at the generic point is either 2 or a power of p, unless $p=2$, in which case it is either 2 or a power of 4.

Chapter II attempts to replace the idea of a limit, in the definition of a derivative in ordinary analysis, by that of specialisation, namely the identical specialisation $u'=u$ of an isomorphism $u\to u'$ of the rational function field $k(x)$; to each place φ in the field $k(x, x')$ is made to correspond a derivation in $k(x)$, which is essentially the same as that obtained by letting x' tend to x as limit when k is a continuous field such as that of complex numbers.

Chapter III first defines the characteristic "cycle" of a one-parameter family of divisors, essentially as the intersection of the generic divisor f of the family with Df, where D is a suitable derivation. Then for a d-parameter family of divisors on a d-dimensional variety it defines a d-dimensional linear system of characteristic cycles (of dimension $d-2$) on the generic member of the family, with a zero-dimensional base cycle, the analogue of the characteristic set of points in classical differential geometry. This is all found to be comparatively little dependent on the characteristic, and closely analogous to what happens in characteristic 0.

In chapter IV, envelopes, regarded as loci touched everywhere by members of the family, are examined. (A formal definition is of course given, but this is the guiding idea.) The situation here is entirely different from that in the case $p=0$. A given family has in general an infinity of envelopes, none of which has any preference over the others; and the cycle of contact of the envelope with the generic member of the family seems to bear very little if any relation to the characteristic cycle on the latter.

In chapter V, the previous results are applied to the problem (which seems to have been the initial motivation of the whole study) of determining all non-ruled surfaces in m-dimensional space which have an $(m-2)$-dimensional family of reducible hyperplane sections. For $p\neq 2$ it is proved that these are exclusively the Veronese surface, and its projections into four and three dimensions (the Kroneker-Castelnuovo theorem, classical for $p=0$). For $p=2$ however the proof breaks down; on the other hand no counterexample is known; thus it remains uncertain whether the theorem is true in this case or not.

P. Du Val (London)

6807:

Samuel, Pierre. Relations d'équivalence en géométrie algébrique. Proc. Internat. Congress Math. 1958, pp. 470–487. Cambridge Univ. Press, New York, 1960.

In this paper, various equivalence relations between cycles on non-singular projective varieties are discussed.

An equivalence relation $\sim$ is called here "adequate" if it is defined between cycles on all non-singular varieties V, W, etc. in projective space and if it satisfies the following conditions: (I) It is compatible with the operation of addition. (II) When X is a cycle on V and the U_j are given subvarieties of V, in finite number, there is a V-cycle X' such that $X \sim X'$ and such that $X' \cdot U_j$ is defined for all j. (III) When X is a V-cycle, and Z a $V \times W$-cycle such that $Z(X) = \mathrm{pr}_W((X \times W) \cdot Z)$ is defined, $X \sim 0$ on V implies $Z(X) \sim 0$ on W. From these three axioms, various properties of equivalence relations are deduced. As examples, rational equivalence, algebraic equivalence, numerical equivalence, square equivalence, n-cube equivalence, abelian equivalence, and pseudo-equivalence are discussed, together with the effect of monoidal transformations on equivalence relations. In the last paragraph, problems are discussed concerning equivalence relations for zero cycles.

Sometimes, an adequate equivalence relation satisfies the following condition: (IV) If X is a V-cycle, (V', X') is a specialization of (V, X) over a field, and V' is again non-singular, then $X \sim 0$ implies $X' \sim 0$. The author states that numerical equivalence and abelian equivalence satisfy (IV), but as far as the reviewer knows, these seem to be unsettled questions (as for numerical equivalence, nothing seems to be known even in the classical case, except for divisors; as for abelian equivalence, we do not seem to know the invariance of irregularities under specializations in the abstract case, which would follow if it satisfied (IV)). If these were settled recently, references should have been given. *T. Matsusaka* (Princeton, N.J.)

6808a:

Ishida, Makoto. On zeta-functions and L-series of algebraic varieties. Proc. Japan Acad. **34** (1958), 1–5.

6808b:

Ishida, Makoto. On zeta-functions and L-series of algebraic varieties. II. Proc. Japan Acad. **34** (1958), 395–399.

6808c:

Ishida, Makoto. Remarks on my previous paper on congruence zeta-functions. Proc. Japan Acad. **35** (1959), 321–322.

In this series of papers, known facts on zeta-functions of abelian varieties are extended to varieties having an abelian variety as a Galois covering, and to L-functions attached to such coverings; some of the existing conjectures on zeta-functions and L-functions can be verified in that case. Part of the author's results are restricted to the case in which the Galois group of the covering is abelian. *A. Weil* (Princeton, N.J.)

6809:

Ishida, Makoto. On congruence L-series. J. Math. Soc. Japan **12** (1960), 22–33.

Earlier conjectures, chiefly due to S. Lang [Bull. Soc. Math. France **84** (1956), 385–407; MR **19**, 578] are further considered, in the light of results due to Y. Taniyama [Sci. Papers Coll. Gen. Ed. Univ. Tokyo **8** (1958), 123–137; MR **21** #4958]. One should note, however, that the author has chosen to modify Lang's definition of an L-series (loc. cit.); it is asserted that both definitions are equivalent for unramified coverings of non-singular varieties. The main result says in substance that Lang's conjecture on the "first layer" of zeros of the L-function for such coverings would follow from a certain assumption on families of "regular" varieties (which amounts to saying that Lang's conjecture is "uniformly" true for a suitable family of such varieties). *A. Weil* (Princeton, N.J.)

6810:

Matsusaka, T. The polarization of algebraic varieties, and some of its applications. Proc. Internat. Congress Math. 1958, pp. 450–453. Cambridge Univ. Press, New York, 1960.

A brief exposition of the main results achieved by the author in his work on polarized varieties and their groups of automorphisms. These have been published in full in three papers in Amer. J. Math. **80** (1958), 45–82, 784–800 and in Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 1–19 [MR **20** #878, 3867; **21** #7213]. *A. Weil* (Princeton, N.J.)

6811:

Lang, Serge; Serre, Jean-Pierre. Erratum à l'article "Sur les revêtements non ramifiés des variétés algébriques". Amer. J. Math. **81** (1959), 279–280.

On désigne par $f: U \rightarrow V$ un revêtement d'une variété algébrique V, par V' une sous-variété irréductible de V, par U_i' les composantes de $f^{-1}(V')$, et par $[U_i' : V']_s$ les facteurs séparables des degrés $[U_i' : V']$. Dans l'article [Amer. J. Math. **79** (1957), 319–330; MR **19**, 320] l'inégalité correcte $\sum [U_i' : V']_s \leq [U : V]$ a été remplacée par (1) $\sum [U_i' : V'] \leq [U : V]$. Cela n'a aucune conséquence pour la suite de l'article. La formule (1) est vraie si V' est simple sur V; un contre-exemple montre qu'il n'en est pas toujours ainsi. *P. Dolbeault* (Poitiers)

6812:

Koizumi, Shoji; Shimura, Goro. On specializations of abelian varieties. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **9** (1959), 187–211.

Soient k un corps, $\mathfrak{o}$ un anneau de valuation discrète de k, $\mathfrak{p}$ son idéal maximal, et $\tilde{k}$ le corps résiduel de k. On ne considère ici que des k-variétés V qu'on peut réduire modulo $\mathfrak{p}$, par exemple des variétés affines ou projectives [cf. Shimura, Amer. J. Math. **77** (1955), 134–176; MR **16**, 616]; on note $\tilde{V}$ le cycle réduction de V; on dit que V est $\mathfrak{p}$-simple si $\tilde{V}$ se réduit à une variété. Étant données deux variétés $\mathfrak{p}$-simples V, W et une application rationnelle $f: V \rightarrow W$, on définit une application rationnelle $\tilde{f}: \tilde{V} \rightarrow \tilde{W}$; étude des points où $\tilde{f}$ n'est pas définie. Une variété de groupe G est dite sans défaut pour $\mathfrak{p}$ si elle est $\mathfrak{p}$-simple et si les applications rationnelles $\tilde{G} \times \tilde{G} \rightarrow \tilde{G}$ et $\tilde{G} \rightarrow \tilde{G}$ déduites de la multiplication et du passage à l'inverse sont partout définies. Étude des anneaux locaux d'un point simple de V et du point correspondant de $\tilde{V}$. Soit L un sous k-espace vectoriel du corps des fonctions rationnelles sur V définies sur k; l'ensemble des $\tilde{f}$ ou $f \in L$ est un $\tilde{k}$-espace vectoriel $\tilde{L}$ de même dimension que L; on en déduit que, si X est un diviseur sur V (supposée complète), on a $l(X) \leq l(\tilde{X})$. Soit A une variété abélienne définie sur k; supposons qu'il existe une extension k' de k, une

extension $(\mathfrak{o}', \mathfrak{p}')$ de la valuation $(\mathfrak{o}, \mathfrak{p})$ à k', et une variété abélienne A' définie sur k', sans défaut pour $\mathfrak{p}'$, et k'-isomorphe à A ; moyennant une hypothèse supplémentaire, il existe une variété abélienne A_1, définie sur k, sans défaut pour $\mathfrak{p}$, et k-isomorphe à A. A partir d'une variété V munie d'une loi de composition normale sans défaut pour $\mathfrak{p}$, on construit (à la Weil) un groupe algébrique G birationnellement équivalent et sans défaut pour $\mathfrak{p}$ (ceci demande une extension de k). Enfin, soient A, B des variétés abéliennes et $A \to B$ un homomorphisme surjectif définis sur k; si A est sans défaut pour $\mathfrak{p}$, il existe une variété abélienne B_1 sans défaut pour p et k-isomorphe à B. Un appendice traite de la spécialisation des anneaux locaux. *P. Samuel* (Clermont-Ferrand)

6813:

Cartier, Pierre. Isogénies des variétés de groupes. Bull. Soc. Math. France **87** (1959), 191–220.

This paper contains some preparatory material for its author's main result, namely the duality theorem, in characteristic $p \neq 0$, between an abelian variety and its Picard variety [see review below]; it consists of a re-elaboration of known results on the foundations of the theory of group-varieties, and of a new approach to the study of isogenies. All group-varieties are taken over a fixed algebraically closed field K, but particular attention is paid to constructions which can be carried out over arbitrary subfields k of K. Chapter 1 contains several elementary results on homomorphisms of group-varieties; compared with the first treatment of the subject [sect. 2 of Ann. Mat. Pura Appl. (4) **38** (1955), 77–119; MR **17**, 193], the author's exposition offers the advantage of dispensing with the "associated form". Chapter 2 is devoted to the study of isogenies: let α be an isogeny of G onto G', so that $K(G') \subseteq K(G)$ (we are using K in order to simplify notations; the paper switches between K and k). It is well known that α can be decomposed into the separable and purely inseparable parts α_s and α_i; the standard tool for the study of α_s has been its kernel, which is a finite subgroup of G, and is also the Galois group $\mathscr{G}$ of $K(G)$ over $K(G')$; the tool for the study of α_i has been, so far, the algebra $\mathscr{D}$ of the invariant hyperderivations on G which vanish on $K(G')$; in turn, if the exponent of inseparability of α_i is p (or 1), $\mathscr{D}$ becomes the enveloping algebra (over K) of the Lie-algebra of the invariant derivations of G which vanish on $K(G')$; and this algebra, considered as a left K-module and extended over $K(G)$, is also the tool used in the Jacobson theory of purely inseparable field extensions [N. Jacobson, Trans. Amer. Math. Soc. **42** (1937), 206–224] (the word "invariant" rather than "right-invariant" has been used, because this method, when introduced by the reviewer in Rend. Circ. Mat. Palermo (2) 5 (1956), 145–169 [MR **18**, 673], applied only to commutative group-varieties). The novelty in the paper under review consists in the unification of the two methods, by means of the Galois-Jacobson-Bourbaki theorem (for which a proof is supplied). If A is the algebra, over $K(G')$, of the endomorphisms of the left $K(G')$-module $K(G)$, A is the extension over $K(G')$ of an algebra N over K; N is the set of the right-invariant elements of A, and is also the tensor product, over K, of $\mathscr{D} \oplus K$ and of the algebra whose basis is $\mathscr{G}$. Main result: Let M be a subalgebra of N; M is related, in the manner described, to an isogeny β of G (which will then be a right factor of α) if and only if (1) $\gamma^{-1}M\gamma \subseteq M$ for each left translation γ on G, and (2) $\Delta M \subseteq M \otimes M$, where Δ is the coproduct in M, namely: $(\Delta m)(x \otimes y) = m(xy)$ for x, $y \in K(G)$; moreover, the correspondence $M \to \beta$ is 1–1, but for isomorphisms of βG. *I. Barsotti* (Providence, R.I.)

6814:

Cartier, Pierre. Isogenies and duality of abelian varieties. Ann. of Math. (2) **71** (1960), 315–351.

First, a notion of a k-structure on a vector space over a field, then a notion of a group operating on a field and a vector space at the same time, are introduced and studied. In chapter II, a study of an algebraic group G operating on a vector space over the function field of some transformation space for G is made, which was necessary to overcome some difficulties in the next chapter. Chapter III is the main portion of this paper and deals with the duality and the absence of (divisorial) torsion on an abelian variety [for the former, see also paper reviewed below; the latter is a known result].

Let A and B be two abelian varieties, α an isogeny from A to B, d the degree of α and k a common field of definition for A, B and α. If $\equiv$ denotes numerical equivalence, to a divisor $D \equiv 0$ on A is associated the set $\mathfrak{G}(D)$ of maps $\psi_{h,a}$ of the function field L of A such that $\psi_{h,a} = h \cdot f_a$. $\mathfrak{G}(D)$ turns out to be a commutative group under the law

$$\psi_{h,a} \cdot \psi_{h' \cdot a'} = \psi_{h \cdot h'_a, a+a'}.$$

When M is the function field of B, the map $: f \to f \circ \alpha = f^\alpha$ is a monomorphism of M into L, giving L a structure of an M-vector space. Let $\mathscr{E}$ be the ring of endomorphisms of this vector space and $\mathfrak{A}(D)$ the subring of $\mathscr{E}$ consisting of the M-linear operators of L which commute with every element of $\mathfrak{G}(D)$. This $\mathfrak{A}(D)$ is called here the Hasse's algebra associated with α, D. Let $\mathfrak{d}$ be a divisor class on B, $\mathfrak{c} = \alpha^{-1}(\mathfrak{d})$, and D an A-divisor whose class is $\mathfrak{c}$. There is a B-divisor E in $\mathfrak{d}$ and a function f on B such that (i) $\alpha^{-1}(E) = D + (f)$, and that (ii) $t \cdot (f) = \chi(t) \cdot f$, where $t \in \mathfrak{A}(D)$ and χ is a character of $\mathfrak{A}(D)$. Then (i) and (ii) define a one-to-one relation between $\mathfrak{d}$ and χ, and $\mathfrak{d}$ is rational over k' if and only if χ maps $\mathfrak{A}(D)_k$ into k'. Using this and the equality

$$[(D)_k; k] = [L_k : M_k] = \mathfrak{d},$$

the author gets the equality $\nu(\alpha) = \nu({}^t\alpha)$. From this the duality and the absence of torsion follow.

T. Matsusaka (Evanston, Ill.)

6815:

Nishi, Mieo. The Frobenius theorem and the duality theorem on an abelian variety. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 333–350.

This paper treats the duality theorem and the theorem of Frobenius on abelian varieties and settles them simultaneously [the first theorem has been settled by P. Cartier in his Paris thesis; cf. also the paper reviewed above]. Here, the author settles the theorem of Frobenius $\nu(\varphi_X) = l(X)^2$ first on a Jacobian variety, where X is a positive non-degenerate divisor, ν means degree of an endomorphism, and φ_X is a certain endomorphism, onto the Picard variety, related to X; $l(X)$ is the dimension of the complete linear system $|X|$. Two proofs are offered here. The first one is based on the idea of expressing a suitable multiple of a symmetric endomorphism as the sum of four

squares due to Morikawa [cf. H. Morikawa, Nagoya Math. J. **6** (1953), 151–170; MR **15**, 464]. The second proof is based upon complicated computations. Next, the author shows that, when A and B are two abelian varieties of the same dimension, isogeneous to each other, there is a constant $f(A, B)$, depending only on A and B, such that for any surjective homomorphism α of A to B, the equality

$$\nu({}^t\alpha) = p^{f(A,B)}\nu(\alpha)$$

holds. Then the author verifies that $f(A, B) = f(A) - f(B)$, $f(A \times B) = f(A) + f(B)$, by going through Jacobian varieties; here $f(A)$, for instance, is an integer determined uniquely by the relation

$$\nu(\varphi x) = p^{f(A)}l(X)^2.$$

(X is a positive non-degenerate A-divisor, and $f(A)$ is determined independently of X.) By taking a suitable Poincaré divisor T on the product of A with its dual, and by showing that

$$T^{(2n)}/(2n)! = p^{-f(A)},$$

where $n = \dim A$, one sees that $f(A) \leq 0$. Then, using the fact that there is a regular surjective homomorphism γ from a product of Jacobian varieties $\prod J_i$ to A, and analysing the exact sequence

$$0 \to B \to \prod J_i \to A \to 0,$$

the author gets $f(A) = 0$. The analysis of this exact sequence is fairly delicate, though short. (Reviewer's note: In the course of the proof, the author shows that $\nu({}^t\alpha)/\nu(\alpha)$ is a power of the characteristic and hence can be expressed as $p^{f(A,B)}$, but this does not seem to be necessary at all.) *T. Matsusaka* (Princeton, N.J.)

6816:

Sampson, J. H.; Washnitzer, G. Cohomology of monoidal transforms. Ann. of Math. (2) **69** (1959), 605–629.

This paper is a study of the behaviour of cohomology groups of coherent algebraic sheaves under monoidal transformations. Let X be a non-singular irreducible algebraic variety, V a non-singular subvariety, and X^* the monoidal transform of X with centre V. Let $\mathscr{F}$ be a coherent sheaf on X, $\mathscr{F}^*$ its reciprocal image on X^*. Then the authors' main theorem asserts that the natural map $H^q(X, \mathscr{F}) \to H^q(X^*, \mathscr{F}^*)$ is an isomorphism in either of the two following cases: (i) $\mathscr{F}$ is locally free; (ii) $\mathscr{F}$ is the extension by zero of a coherent sheaf on V. An example, due to J.-P. Serre, shows that the theorem would be false if no restriction were placed on $\mathscr{F}$.

The last part of the paper is devoted to the sheaves Tor $(\mathscr{G}, \mathscr{F}^*)$, where $\mathscr{G}$ is coherent on X, $\mathscr{F}^*$ coherent on X^*. A number of potentially useful results are proved, and there is a considerable overlap with the calculations in Grothendieck's proof of the Riemann-Roch theorem [see following review]. *M. F. Atiyah* (Cambridge)

6817:

Borel, Armand; Serre, Jean-Pierre. Le théorème de Riemann-Roch. Bull. Soc. Math. France **86** (1958), 97–136.

The advent of sheaf theory has, amongst many things, brought with it a great development of the classical theorem of Riemann-Roch. This paper is devoted to Grothendieck's version of the theorem. Grothendieck has generalized the theorem to the point where not only is it more generally applicable than the F. Hirzebruch's version [*Neue topologische Methoden in der algebraischen Geometrie*, Springer-Verlag, 1956; MR **18**, p. 509], but it depends on a simpler and more natural proof.

The paper originated from the notes of a seminar devoted to the work of Grothendieck which the authors conducted in Princeton during the fall of 1957, and the opening phrase asserts their essentially editorial role.

The result of this unusual three-way collaboration is a remarkably clear, short, and highly motivated presentation of the Grothendieck theorem. Of necessity, such an exposition is primarily directed at the expert, and the paper is quite hard going for those of us who are not intimately acquainted with their basic references, in particular, with J.-P. Serre's paper "Faisceaux algébriques cohérentes" [Ann. of Math. (2) **61** (1955), 197–278; MR **16**, 953], which unquestionably lays the foundation for Grothendieck's work. In their single-mindedness, the authors have also omitted an introduction, and start off at once with preparatory material towards theorem I.

This first goal of theirs is the following: Let $f: X \to Y$ be a proper map of quasi-projective varieties, let $\mathscr{F}$ be a coherent sheaf on X, and let the sheaves $R^qf(\mathscr{F})$ on Y be defined by $R^qf(\mathscr{F})_U = H^q(f^{-1}(U); \mathscr{F})$ (U open in X). Then these sheaves are also coherent.

This theorem has vital consequences for their study of the group $K(X)$ which they introduce next. If X is an algebraic variety (always over an arbitrary algebraically closed field) the group $K(X)$ is defined as follows. Let $F(X)$ denote the free abelian group generated by coherent sheaves over X. Also, if $E: 0 \to \mathscr{F}_1 \to \mathscr{F} \to \mathscr{F}_2 \to 0$ is a short exact sequence of such sheaves, let $Q(E)$ be the "word" $\mathscr{F} - (\mathscr{F}_1 + \mathscr{F}_2)$ in $F(X)$. Now define $K(X)$ as the quotient of $F(X)$ modulo the subgroup generated by $Q(E)$ as E ranges over the short exact sequences. (We call this construction the K-construction; it can clearly be applied to any category in which short exact sequences are defined.) For example, if p is a point, then $K(p) \approx \mathbf{Z}$ (= the ring of integers), the isomorphism being determined by attaching to a sheaf (which is merely a module over the ground-field in this case) its dimension. This homomorphism is denoted by $ch: K(p) \xrightarrow{\approx} \mathbf{Z}$.

As will be seen, the Riemann-Roch theorem is a comparison statement about $K(X)$ and the Chow ring $A(X)$ which is valid only on non-singular varieties. Accordingly, we will let $\mathfrak{A}$ denote the category of quasi-projective non-singular varieties and their proper maps. On this category $K(X)$ and $A(X)$ partake of both a covariant and a contravariant nature, and it is precisely to complete $K(X)$ to a covariant functor that theorem I is essential.

Grothendieck denotes this covariant homomorphism, induced by a map $f: X \to Y$ in $\mathfrak{A}$, by $f_!$, and defines it in this way: If $\mathscr{F}$ is a sheaf (coherent, algebraic, will be understood hereafter) then $f_!(\mathscr{F}) \in K(Y)$ shall be the class of the word $\sum_q (-1)^q R^qf(\mathscr{F})$ in $K(Y)$. Because the sum is finite on objects in $\mathfrak{A}$ this operation is well defined, and its linear extension to $F(X)$ is seen to vanish on words of the form $Q(E)$, thus inducing a homomorphism $f_!: K(X) \to K(Y)$.

The naturality condition $(f \circ g)_! = f_! \circ g_!$ is valid, and follows from the spectral sequence which relates $R^q(f \circ g)$ to R^pf and R^tg. Thus the obvious "Euler characteristic"

nature of $f_!$ is essential not only for the vanishing of $f_!$ on $Q(E)$, but also for the naturality! Note also that if $f: X \to p$ is the map onto a point, then $ch\, f_!(\mathscr{F})$ may be identified with $\sum(-1)^q \dim H^q(X; \mathscr{F}) = \mathfrak{X}(X; \mathscr{F})$; and it is an expression of this sort which was evaluated by Hirzebruch in his topological version of the Riemann-Roch theorem by a certain cohomology class. In short, $f_!$ is a very "good" notion.

In the Grothendieck theory, the role of cohomology is taken over by the Chow ring $A(X)$, of cycles under linear equivalence, the product being defined by intersection. On our category, $A(X)$ also has a covariant side to it, namely $f_*: A(X) \to A(Y)$, defined by the direct image of a cycle. However, f_* is only an additive homomorphism. The contravariant extension of $A(X)$, i.e., $f \to f^*$ where f^* is induced by the inverse image of a cycle, is of course a ring homomorphism; and these two operations are linked by the permanence law: $f_*((x) \cdot f^*(y)) = f_*(x) \cdot y$, $x \in A(X)$, $y \in A(Y)$.

The contravariant properties of $K(X)$ are best brought out with the aid of the following theorem II: Let $K_1(X)$ be the group obtained by applying the K-construction to the category of algebraic vector bundles over X, $X \in \mathfrak{A}$. Also, let $\varepsilon: K_1(X) \to K(X)$ be the homomorphism defined by the operation which assigns to a bundle the sheaf of germs of its sections. Then ε is a bijection.

To a topologist at least, this theorem is reminiscent of the Poincaré duality theorem. In any case, by identifying $K(X)$ with $K_1(X)$ one may induce the obvious (inverse image of a bundle) contravariant extension of $K_1(X)$ to $K(X)$. This homomorphism is denoted by $f^!$. Further, the ring structure of $K_1(X)$ induced by the tensor product of bundles is now also impressed on $K(X)$, and as the authors show, the permanence relation is again valid: $f_!(x \cdot f^!(y)) = f_!(x) \cdot y$ for a map f in $\mathfrak{A}$. This new interpretation of $K(X)$ (i.e., as $K_1(X)$) brings with it also a ring homomorphism $ch: K(X) \to A(X) \otimes \mathbf{Q}$ which is natural on the contravariant side (namely, $ch(f^{!*}x) = f^* ch(x)$) and agrees with our definition of ch on $K(p)$. This function is derived from the Chern character of bundles and can be characterized by: (1) If L is a line bundle over $X \in \mathfrak{A}$, then $ch(L) = e^c = 1 + c + c^2/2! + \cdots$, etc., where $c = c_1(L)$ is the class in $A(X)$ of the zeros of a generic rational section of L; (2) ch is a ring homomorphism; (3) the naturality condition already recorded. (See the next review.)

In general, the identification of $K_1(X)$ with $K(X)$ extends the notion of characteristic classes from vector bundles to coherent sheaves. We will, in particular, have need of the Todd-class, which on vector bundles is uniquely characterized by these conditions: (1) If L is a line bundle over an object X in $\mathfrak{A}$, then $T(L) = c/(1-e^c)$, where $c = c_1(L)$ as defined earlier; (2) T is multiplicative: $T(E+F) = T(E) \cdot T(F)$; (3) $Tf^! = f^* T$ for maps in $\mathfrak{A}$.

This Todd class enters the answer to the following, in our context very natural, question: How does $ch: K(X) \to A(X) \otimes \mathbf{Q}$ behave under the covariant homomorphisms $f_!$ and f_*? The answer to this question is precisely the Riemann-Roch formula of Grothendieck: (Riemann-Roch theorem). Let f be a map $X \to Y$, in $\mathfrak{A}$. Then

$$ch\{f_!(x)\} \cdot T(Y) = f_*\{ch(x) \cdot T(X)\},$$

where $x \in K(X)$, and $T(X)$, $T(Y)$ denote the values of the Todd class on the tangent bundles of X and Y respectively.

The Hirzebruch formula is an immediate corollary; just let f be the projection onto a point, and let x be represented by a locally free sheaf $\mathscr{F}$. Then the left-hand side reduces to $\mathfrak{X}(X; \mathscr{F})$ as remarked earlier, while the right-hand side gives the coefficient of $ch(\mathscr{F}) \cdot T(X)$ in the dimension of X, which Hirzebruch denotes by $\kappa_n(ch(\mathscr{F}) \cdot T(X))$.

The great advantage of Grothendieck's formulation is its dynamic nature. This enables one to prove the general theorem by considering special situations. Notably one concludes by the graph-construction that it is sufficient to prove the Riemann-Roch theorem in the following two cases: (a) $f: Y \times P \to Y$ is the projection onto Y, P being a projective space; (b) $f: Y \to X$ is a closed imbedding. These are then treated by quite different methods. To prove (a), the authors first prove a Künneth type theorem to the effect that $K(X) \otimes K(P) \to K(X \times P)$ is surjective. This fact, together with the Riemann-Roch formula for the projection of P onto a point—which is checked explicity—proves (a). To establish (b), the authors first treat a special case of Riemann-Roch theorem for Y a divisor on X. This special case is quite simple and at the same time illuminating in that it essentially forces the Todd class upon one, once one seeks a formula for the extent to which ch and $f_!$ fail to commute. Here is the gist of the argument. Assume that $i: Y \subset X$ is a regular divisor of X, and let L be the line bundle it determines. Thus $c_1(L) = Y$ and $L|Y$ is the normal bundle of Y in X. We propose to compute both $ch\{i_!(y)\}$ and $i_*\{ch(y)\}$ and see by how much they differ when $y \in K(Y)$ is the class of the structure sheaf O_Y, or, interpreted in $K_1(Y)$, when Y is the class of the trivial bundle 1. Let $S(L^{-1})$ be the sheaf of germs of sections of L^{-1}. Then multiplication with a regular section in L which vanishes on Y gives rise to the exact sequence of sheaves

$$0 \to S(L^{-1}) \to O_X \to \hat{O}_Y \to 0,$$

where $\hat{O}_Y$ is the structure sheaf of Y trivially extended to X. Now one first verifies that $i_!(O_Y)$ is represented by $\hat{O}_Y$. It therefore follows from our exact sequence that $i_!(1) = 1 - L^{-1}$ (using the $K_1(X)$ version of $K(X)$), whence $ch(i_!(1)) = 1 - e^{-Y}$. On the other hand, $ch(1) = 1$, whence $i_*\{ch(1)\} = Y$. So then

$$ch(i_!1) = i_*\{ch(1)\} \cdot T(L)^{-1} = i_*(T(i^!L)^{-1}),$$

the last step following from the permanence relation. This expression is equivalent to the Riemann-Roch formula with $y = 1$. Indeed if we multiply both sides by $T(X)$, use the permanence again on the right-hand side, and recall that $i^*T(X) = T(Y) \cdot T(i^!L)$ (because $i^!L$ is the normal bundle to Y and T is multiplicative), the above goes into $ch(i_!(1)T(X)) = i_*\{T(Y)\}$, which is just the special Riemann-Roch formula with X and Y reversed. Thus if a formula of the type we are seeking is at all possible, then the correction term will have to satisfy the axioms which were prescribed for T.

To complete the case of an imbedding $Y \subset X$, the authors blow up X along Y to obtain a new object X' in $\mathfrak{A}$, together with a projection $f: X' \to X$. The inverse image of Y under f' is then a regular divisor Y' of X', and by a series of ingenious arguments, the Riemann-Roch theorem for f_* is now reduced to the special Riemann-Roch theorem for the injection $Y' \to X'$. The reduction is in a sense the most difficult and certainly the most detailed step in the paper.

This then is a rough plan of the proof, and the methods of Serre [loc. cit.] essentially suffice to carry out the program. There are occasions, however, where the more abstract homological algebra of a previous paper by Grothendieck [Tôhoku Math. J. (2) **9** (1957), 119–221; MR **21** #1328] is useful.

Although the paper pursues its goal relentlessly, it is nevertheless so rich in ideas and auxiliary results which are clearly more generally applicable, that I will not even try to do justice to them.

It seems to me appropriate to close this review with a word of thanks to the authors for presenting us with such an informal and tightly knit account of so many interesting ideas. To my mind, this is the best method of mathematical communication. Also, an account of this sort was especially needed in view of Grothendieck's chilling announcement which puts the topics discussed here into Chapter 12—if we start counting with 1—of his already bulging foundation [Inst. Hautes Études Sci. Publ. Math. No. 4 (1960)]. *R. Bott* (Cambridge, Mass.)

6818:

Grothendieck, Alexander. La théorie des classes de Chern. Bull. Soc. Math. France **86** (1958), 137–154.

This discussion of the Chern class forms a sort of appendix to the paper on the Riemann-Roch theorem by Borel and Serre [see preceding review], and brings among other things a definition of the Chern class so general that it contains, as special cases, all the usual different constructions. The author postulates a functor $X \to A(X)$, from a category $\mathscr{V}$ of non-singular algebraic varieties and their morphisms to graded anticommutative modules, which has all the general properties of the cohomology functor. He then gives axioms on $\mathscr{V}$ and A which enable him to define the Chern class of a vector bundle over X as an element of $A(X)$.

This review will only discuss the author's construction when $\mathscr{V}$ is the category of quasi-projective nonsingular algebraic varieties, and $A(X)$ is the ring of classes of cycles on X under rational equivalence. Thus $A(X)$ is the Chow ring, $\sum A^{2p}(X)$, with $A^{2p}(X)$ denoting the classes of cycles of codimension p.

The essential link between an (algebraic) vector bundle E, over X, and $A(X)$ is furnished by the line bundles. The author denotes the group of line bundles over X (under the tensor product) by $P(X)$; then the operation which assigns to L the zeroes of a rational section of L (which does not vanish identically on any component) defines a homomorphism $p_x: P(X) \to A^2(X)$ (which in this case is actually a bijection). Granted this homomorphism, the Chern class is defined in this manner: If E is a vector bundle, of dimension p over X, then $P(E)$ shall denote the bundle of lines in E. Thus $P(E)$ is fibered over X by projective spaces. The functor $X \to P(E)$ preserves the category $\mathscr{V}$. Furthermore, there is a canonical line bundle $\check{L}_E$ on $P(E)$ determined tautologically: let (x, l) be a point of $P(E)$; thus l is a line in the fiber E_x; now the fiber of $\check{L}_E$ above (x, l) is by definition the line l itself. The dual bundle to $\check{L}_E$ is denoted by L_E. This construction is then so arranged that the restriction of L_E to any fiber of the projection $f: P(E) \to X$, is the hyperplane bundle. Now the author shows that there are uniquely defined elements $c_i(E) \in A^{2i}(X)$ which satisfy the following equation in $A(P(E))$:

$$\sum_{i=0} f^* c_i(E)(\xi_E)^{p-i} = 0,$$

where $\xi_E = p_x(L_E) \in A^2(P(E))$, while $C_0(E) = 1$. By defining $c_i(E) = 0$ for $i > p$, the element $1 + \sum_1^\infty c_i(E)$ is therefore a well determined element of $A(X)$, and this is the Chern class $c(E)$. {In principle this approach is, I think, due to Wu who used it for the Stiefel-Whitney classes. In the algebraic case essentially the same construction has been given by Segre [Ann. Mat. Pura Appl. (4) **35** (1953), 1–27; MR **15**, 822].}

The author goes on to prove the usual properties of the Chern class and shows that they serve to characterize it. These are the following. (1) If $f: X \to Y$ is a morphism, then $c\{f^{-1}(E)\} = f^*\{c(E)\}$. (2) If E is a line bundle, then $c(E) = 1 + p_x(E)$. (3) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence, then $c(E) = c(E') \cdot c(E'')$.

As was already mentioned, the author discusses this construction in considerably greater generality than I have recounted here. By axiomatising all the properties of $A(X)$ vis-a-vis the category $\mathscr{V}$, which he needs, his proofs are seen to apply to various definitions of the Chern class, notably to the Atiyah definition involving holomorphic differential forms in the complex case.

On the other hand, the axioms on $A(X)$ are quite potent, the most powerful one being the one which asserts that $A\{P(E)\}$ is additively freely generated by $1, \xi_E, \cdots, \xi_E^{p-1}$ over $A(X)$. (The ring $A\{P(E)\}$ is a module over $A(X)$ by virtue of the projection $f: P(E) \to X$.) Once this axiom is granted, the "splitting method" achieves the rest: one considers $D(E)$, the bundle of flags in E; again tautologically, the bundle E splits entirely when lifted to this variety. Here then one may, via p_x, get a hold on the line bundles which make up the lifted bundle. All this is very clearly and elegantly carried out in the first part of this note.

The latter part is devoted to functorial properties of the completed Chern class. If $K(X)$ denotes the effect of the K-construction on the category of algebraic vector bundles over X (see previous review), then the completed Chern class is interpreted as a ring-homomorphism of $K(X)$ into a ring $\widetilde{A(X)}$ constructed out of $A(X)$ as follows. Let $\hat{A}(X)$ be the direct product of the $A^i(X)$. Then the elements of $\hat{A}(X)$ whose first component is 1 form a group under multiplication, to be denoted by $1 + \hat{A}(X)^+$. In the product group $Z \times (1 + \hat{A}(X))$ (which is written additively) the author defines a product structure, and denotes the resulting ring by $\widetilde{A(X)}$. Now if E is a bundle over X, its dimension $rg(E)$ defines, by linear extension, a ring-homomorphism of $K(X)$ into the integers $\mathbf{Z}$. The completed Chern class is the homomorphism $\tilde{c}: K(X) \to \widetilde{A(X)}$ defined by: $\tilde{c}(E) = (rg(E), c(E))$. Clearly $\tilde{c}$ is an additive homomorphism; it turns out to be a ring-homomorphism by definition, as the ring structure in $\widetilde{A(X)}$ is defined precisely with this in view: if E and E' are two bundles, then $c(E \otimes E')$ is a well determined function of $c(E)$, $c(E')$; and dim (E), dim (E'), in fact the polynomials which express $c_i(E \otimes E')$ in terms of the $c_i(E)$, $c_i(E')$, $rg(E)$, $rg(E')$, have integral coefficients, and the author defines the ring $\widetilde{A(X)}$ precisely in terms of

these universal polynomials. (The step from $A(X)$ to $\widetilde{A(X)}$ turns out to be a well determined operation on graded modules in general.) In a similar manner (that is, by analogy from $K(X)$) the author introduces exterior power operators λ^i into $\widetilde{A(X)}$, and remarks that with the aid of this notion a refinement of his Riemann-Roch theorem (see preceding review) with regard to torsion is possible, but so far only when the variety is defined over a field of characteristic 0. The author next compares $K(X)$ and $\widetilde{A(X)}$; in particular he shows with the aid of his Riemann-Roch theorem that, modulo torsion, $\tilde{c}:K(X)\to\widetilde{A(X)}$ is a bijection.

The paper closes with a discussion of the zeroes of a regular section in a vector bundle.

R. Bott (Cambridge, Mass.)

LINEAR ALGEBRA

6819:

Rau, P. S. On positive definite quadratic forms. Math. Student **26** (1958), 165–168.

The proof presented here of Frobenius's determinantal criterion for positive definiteness of a quadratic form does not seem to offer any obvious advantages over the existing proofs of the theorem in question.

L. Mirsky (Sheffield)

6820:

Marcus, Marvin. On a determinantal inequality. Amer. Math. Monthly **65** (1958), 266–268.

Let λ_j, α_j, β_j be the eigenvalues of $I-A^*B$, A^*A and B^*B so indexed that $|\lambda_j|\geq|\lambda_{j+1}|$, $\alpha_j\geq\alpha_{j+1}$, $\beta_j\geq\beta_{j+1}$ for $j=1,\cdots,n-1$, where A and B are n-rowed matrices and A^* denotes the conjugate transpose of A. The author proves that, if $\alpha_n\geq 0$, $\beta_n\geq 0$, then for each k satisfying $1\leq k\leq n$ we have

$$\prod_{j=1}^{k}|\lambda_{n-j+1}|^2\geq\prod_{j=1}^{k}(1-\alpha_j)(1-\beta_j).$$

In case $k=n$, the inequality is due to the reviewer.

L. K. Hua (Zbl **83**, 8)

6821:

Fiedler, Miroslav. A remark on positive definite matrices. Časopis Pěst. Mat. **85** (1960), 75–77. (Czech. Russian and English summaries)

Author's summary: "For a square matrix A let A^* be the conjugate transpose, $\tau(A)$ the trace, $N(A)=\tau(AA^*)^{1/2}$, $M(A)$ the square root of the maximal eigenvalue of AA^*. The following theorem is proved. Let A, B be positive definite matrices of the same order. Then $\tau[(A-B)\times(B^{-1}-A^{-1})]\geq N(A-B)^2M(A)^{-1}M(B)^{-1}$. Some consequences of this inequality are given, e.g.: A positive definite matrix is uniquely determined by its elements on some places of its scheme and by the elements of its inverse matrix on the remaining places."

H. Schwerdtfeger (Montreal)

6822:

Bauer, F. L.; Householder, A. S. Some inequalities involving the euclidean condition of a matrix. Numer. Math. **2** (1960), 308–311.

For a non-singular $n\times n$-matrix A let $\|A\|=\sigma_{\max}(A)$ be the norm associated with the euclidean vector norm, i.e., the square root of the greatest eigenvalue of A^HA, $A^H=A^{-1}$; then $\kappa(A)=\|A\|\cdot\|A^{-1}\|=\sigma_{\max}(A)/\sigma_{\min}(A)$ is called the euclidean condition of A. If now ξ and η are two unit n-vectors, $\xi^H\eta=\exp(i\Phi)\cos\varphi$ $(0\leq\varphi\leq\pi/2)$, if ζ is a unit vector in the plane of ξ, η orthogonal to ξ, and if Q is a 2×2-matrix such that $(\xi\eta)=(\xi\zeta)Q$, then with $\bar{M}=(A\xi A\eta)^H(A\xi A\eta)$, $\tilde{M}=Q^H\bar{M}Q$ one has $\kappa(\tilde{M})\leq\kappa(Q)^2\kappa(\bar{M})=\cot^2\frac{1}{2}\varphi\cdot\kappa(\bar{M})\leq\kappa(A^HA)\cot^2\frac{1}{2}\varphi$. A second inequality is obtained from the geometric-arithmetic mean inequality for the diagonal elements $\xi^HA^HA\xi$, $\eta^HA^HA\eta$ of the matrix $\bar{M}$. Two more inequalities referring to more complicated conditions are too difficult to explain.

H. Schwerdtfeger (Montreal)

6823:

Carlitz, L.; Levine, Jack. An identity of Cayley. Amer. Math. Monthly **67** (1960), 571–573.

The following theorem is proved. Let $|a_{ij}|$ be a determinant of order n and let $^+|a_{ij}|^+$ denote the corresponding permanent. If the rank of the matrix $[a_{ij}]$ is less than or equal to 2 and if all $a_{ij}\neq 0$, then $^+|1/a_{ij}|^+\cdot|1/a_{ij}|=|1/a_{ij}^2|$. The case $n=3$ was first proved by Cayley, and the case $n=4$ was proved by Levine [same Monthly **66** (1959), 290–292; MR **21** #3442].

J. K. Goldhaber (College Park, Md.)

6824:

Helfand, Eugene. On inversion of the linear laws of irreversible thermodynamics. J. Chem. Phys. **33** (1960), 319–322.

The mathematical problem considered is the inversion of a real symmetric linear transformation on a set of linearly dependent vectors. By considering the projection of the transformation in the subspace in which it is non-singular, an "inverse" transformation is constructed in terms of its eigenvectors (and associated eigenvalues) in this subspace. {A reference to E. H. Moore [*General analysis*, v. I, Amer. Philos. Soc. Philadelphia, 1935] would have been appropriate.} Application is made to the equations of irreversible thermodynamics.

D. Falkoff (Waltham, Mass.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 6797, 6803.

6825:

Mori, Yoshiro. On the integral closure of an integral domain. VI. On the notion of Artin's symbols. III. Bull. Kyoto Gakugei Univ. Ser. B **15** (1959), 14–16.

Dans un anneau intègre A, un F-idéal est un idéal $\mathfrak{f}$ qui vérifie $\mathfrak{f}^{-1}\mathfrak{f}=\mathfrak{f}$ [cf. Mori, même Bull. **13** (1958), 1–3; **14** (1959), 1–3; MR **21** #3447, 7223]. Étude des suites de composition formées de F-idéaux, et de la décomposition des F-idéaux en produits. *P. Samuel* (Clermont-Ferrand)

6826:

Sussman, Irving. Ideal structure and semigroup domain decomposition of associate rings. Math. Ann. **140** (1960), 87–93.

The author has defined earlier [Math. Ann. **136** (1958), 326–338; MR **20** #6993] an associate ring to be a ring R such that (i) R is a subdirect product of rings R_i with identity and without zero divisors, (ii) $a \in R$ implies that $a^0 \in R$, where the ith component of a^0 is 1_i if $a_i \neq 0$ and 0 if $a_i = 0$. Here it is shown that if the boolean algebra J of all idempotent elements of a regular ring R is atomistic, then R is a subdirect sum of all the ideals Re with e an atom of J.
B. Jónsson (Minneapolis, Minn.)

6827:

Subrahmanyam, N. V. Lattice theory for certain classes of rings. Math. Ann. **139**, 275–286 (1960).

The following two properties of a ring R are considered: (P_1) For each $a \in R$ there exists a central idempotent $a^0 \in R$ such that $aa^0 = a^0$ and, for any idempotent $e \in R$, $ae = ea = a$ implies $ea^0 = a^0$. (P_2) For each $a \in R$ there exists a central idempotent $a_0 \in R$ such that $aa_0 = 0$ and, for any idempotent $e \in R$, $ae = ea = 0$ implies $ea_0 = a_0$. The condition (P_1) generalizes Sussman's notion of an associate ring [Math. Ann. **136** (1958), 326–338; MR **20** #6993]. In a ring with identity (P_1) and (P_2) are equivalent, but in a ring without identity they are never both satisfied. In a (P_1) ring a partial ordering is introduced, $a < b$ if $a^0 b = a$, and a relation of compatibility, $a \sim b$ if $a^0 b = ba^0$. The principal results concern maximal sets M of compatible elements. It is shown that each such set is a distributive lattice under $<$, with $a \cup b = a + b - a^0 b$ and $a \cap b = ab^0$. One of these sets is R_0, the set of all idempotents (every idempotent is in the center); and every $a \in R$ belongs to a maximal set that is lattice isomorphic to R_0. These results are applied to a (P_2) ring R by showing that R can be embedded in a (P_2) ring R' with identity, which is therefore also a (P_1) ring, in such a way that every maximal set M of compatible elements in R is also maximal in R'.
B. Jónsson (Minneapolis, Minn.)

6828:

Sussman, Irving; Foster, Alfred L. On rings in which $a^{n(a)} = a$. Math. Ann. **140** (1960), 324–333.

An SP ring (simply periodic ring) is a ring in which every element satisfies an equation $a^{n(a)} = a$ with $n(a) \geq 2$. An SPB ring is an SP ring in which the exponents $n(a)$ are bounded. The principal results are: (I) A ring with identity is an SP ring if and only if it has no nilpotent element and some power of every element is idempotent. (II) An SP ring can be uniquely partitioned into sets that are groups under the ring multiplication. (III) Every SPB ring is a subdirect sum of fields whose characteristics are different from zero and have a common upper bound.
B. Jónsson (Minneapolis, Minn.)

6829:

Mori, Yoshiro. On the fundamental theorem of regular local rings. Bull. Kyoto Gakugei Univ. Ser. B **15** (1959), 17–22.

L'auteur démontre, sans homologie, que tout anneau local régulier R est factoriel; la difficulté du cas où R est ramifié est surmontée au moyen d'un lemme sur les discriminants. On voit en chemin que, si le complété $\hat{R}$ d'un anneau local noethérien R est factoriel, R est aussi factoriel. Exemple d'un anneau local noethérien factoriel dont le complété n'est pas factoriel. Si un anneau local noethérien R de dimension r admet une chaine $(0) = \mathfrak{p}_0 < \mathfrak{p}_1 < \cdots < \mathfrak{p}_{r-1}$ d'idéaux premiers non maximaux tels que les $R/\mathfrak{p}_i$ soient factoriels, alors R est régulier.
P. Samuel (Clermont-Ferrand)

6830:

Harada, Manabu. Note on raising idempotents. J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 63–65.

Let R be a ring, I an ideal with $\bigcap I^n = 0$; when $A \subset R$, let A^Δ denote the completion of A in the I-adic topology. If $(I^n)^\Delta = (I^\Delta)^n$ for $n = 1, 2, \cdots$, then finite sets of orthogonal idempotents can be raised from R/I into R^Δ. This condition on I is satisfied if I is finitely generated both as a left and as a right R-module. If, besides, R/I satisfies the minimum condition, then R^Δ decomposes correspondingly.
D. Zelinsky (Evanston, Ill.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

6831:

Albert, A. A. Finite division algebras and finite planes. Proc. Sympos. Appl. Math., Vol. 10, pp. 53–70. American Mathematical Society, Providence, R.I., 1960.

Une division-algebra $\mathfrak{D}$ est une algèbre dans laquelle l'ensemble $\mathfrak{D}^*$ des éléments non nuls est un quasigroupe avec élément neutre par rapport à la multiplication. Si $\mathfrak{D}$ est seulement une algèbre sans diviseurs de zéro, alors $\mathfrak{D}^*$ est un quasigroupe. Après une série de lemmes concernant les isotopies d'une division-algebra, et les algèbres définies sur un corps fini, en particulier celles que l'on peut construire à partir du corps du second ordre, la connexion entre chaque division-algebra finie non associative $\mathfrak{D}$ et le plan projectif fini $\mathfrak{M}(\mathfrak{D})$ qu'elle définit est discuté. Condition pour que $\mathfrak{M}(\mathfrak{D})$ soit arguésien. Si deux algèbres $\mathfrak{D}$ et $\mathfrak{D}'$ ont le même nombre d'éléments, et si $\mathfrak{D}$ n'est pas associative, alors, tout isomorphisme de $\mathfrak{M}(\mathfrak{D})$ avec $\mathfrak{M}(\mathfrak{D}')$ projette les éléments à l'infini de $\mathfrak{M}(\mathfrak{D})$ sur ceux de $\mathfrak{M}(\mathfrak{D}')$. Si $\mathfrak{D}$ et $\mathfrak{D}'$ sont deux algèbres à division finies et si $\mathfrak{D}$ n'est pas associative, alors pour que les plans projectifs finis correspondants soient isomorphes il faut et il suffit que $\mathfrak{D}$ et $\mathfrak{D}'$ coïncident par une isotopie de composantes (non singulières) linéaires dans un corps premier $\mathfrak{F}_p$.
A. Sade (Marseille)

6832:

Leadley, J. D.; Ritchie, R. W. Conditions for the power associativity of algebras. Proc. Amer. Math. Soc. **11** (1960), 399–405.

Results of Albert [Summa Brasil. Math. **2** (1948), no. 2, 21–32; MR **10**, 97] and Kokoris [Trans. Amer. Math. Soc. **77** (1954), 363–373; same Proc. **6** (1955), 705–710; MR **16**, 442; **17**, 342], giving conditions for power-associativity of commutative rings, are generalized to noncommutative algebras. It is proved that if A is an algebra over a field of characteristic $p \neq 2, 3, 5$ and if the identities $x^2x = xx^2$, $x^3x = x^2x^2$, $x^{p^r-1}x = xx^{p^r-1}$ hold for every positive integer r, then A is power-associative. If the characteristic is 5 then $x^2x = xx^2$, $x^3x = x^2x^2$, $x^5x = x^4x^2$

and $x^{5^r-1}x = xx^{5^r-1}$ for all x in A and all positive integers r imply that A is power-associative. When the base field has characteristic 3 and is not the prime field, $x^3x = x^2x^2$, $x^4x = x^3x^2$ and $x^{3^r-1}x = xx^{3^r-1}$ for all x and r imply that A is power-associative. If the base field has characteristic 2 and is not the prime field, then A is power-associative if $x^2x = xx^2$, $x^3x = x^2x^2 = xx^3$ and $x^{2^r-1}x = x^{2^{r-1}}x^{2^{r-1}}$ for all x, r. Examples are given to show that these hypotheses are the best possible. *L. A. Kokoris* (Chicago, Ill.)

6833:

Boers, A. H. Quelques remarques par rapport a l'anneau assosymétrique. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 192–195.

Im Anschluss an eine Arbeit von E. Kleinfeld [Proc. Amer. Math. Soc. **8** (1957), 983–986; MR **19**, 726] über assosymmetrische Ringe und an eigene Untersuchungen [Nederl. Akad. Wetensch. Proc. Ser. A **59** (1956), 532–534; MR **20** #5227] beweist Verf., dass ein assosymmetrischer Ring 5-associativ ist, wenn die Charakteristik von 2 und 3 verschieden ist, und dass ein assosymmetrischer kommutativer Ring associativ ist. Ferner beweist er in Verschärfung eines früheren Resultates, dass ein 4-associativer Ring, dessen Charakteristik von 2 und 3 verschieden ist, associativ ist, wenn er kein Ideal J enthält mit $J \neq 0$ und $J^2 = 0$.

R. Moufang (Frankfurt a.M.)

6834:

Kokoris, Louis A. Nodal non-commutative Jordan algebras. Canad. J. Math. **12** (1960), 488–492.

The author determines the structure of finite-dimensional simple power-associative algebras A (over a field F of characteristic $\neq 2$) which are nodal (i.e., every element has the form $\alpha 1 + z$ with α in F and z nilpotent, but A cannot be written as $F1 + N$ with N a nil subalgebra of A; the terminology is that of R. D. Schafer, Proc. Amer. Math. Soc. **9** (1958), 110–117 [MR **21** #2677]) and also are non-commutative Jordan algebras (i.e., A is flexible, and the algebra A^+ with product $u \cdot v = \frac{1}{2}(uv + vu)$ is a Jordan algebra).

The result is that $A = F1 + N$, where N^+ is a commutative associative (truncated) polynomial ring over F in n variables: $N^+ = F[x_1, \cdots, x_n]$, $x_i^p = 0$, where p (necessarily $\neq 0$) is the characteristic of F. The x_i can be selected to satisfy: $x_ix_j = a_{ij}1 + w_{ij}$, a_{ij} in F, w_{ij} in N, and for each i some $a_{ij} \neq 0$. The product in A is given by:

$$f(x_1, \cdots, x_n)g(x_1, \cdots, x_n) = f \cdot g + \tfrac{1}{2} \sum_{i,j} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot [x_i, x_j]$$

(where $u \cdot v$ denotes $\frac{1}{2}(uv + vu)$ and $[u, v]$ denotes $uv - vu$).

B. Harris (Princeton, N.J.)

HOMOLOGICAL ALGEBRA

6835:

Kuniyoshi, Hideo. Cohomology theory and different. Tôhoku Math. J. (2) **10** (1958), 313–337.

Let R be a Dedekind ring, K its quotient field, L a finite separable extension field and Λ the principal order of L over R, considered as an algebra over R. For any two-sided Λ-module A the homology and cohomology groups are defined as in Cartan and Eilenberg, *Homological algebra* [Princeton Univ. Press, Princeton, N.J., 1956; MR **17**, 1040; Chapter IX]. Namely, setting $\Lambda^e = \Lambda \otimes \Lambda$, $H_n(\Lambda, A) = \mathrm{Tor}_n^{\Lambda^e}(A, \Lambda)$, and $H^n(\Lambda, A) = \mathrm{Ext}_{\Lambda^e}^n(A, \Lambda)$. The author defines the left n-homological different $D_n^l(\Lambda/R)$ as $\{\lambda \in \Lambda \mid \lambda \tilde{\otimes} 1\, H_n(\Lambda, A) = 0$ for all $A\}$. He defines left n-cohomological different, right n-homo- and cohomological differents, two sided n-homo- and cohomological differents, and commutative n-homo- and cohomological differents in a similar way. These definitions are a generalization of the definition of different given by Y. Kawada [Ann. of Math. (2) **54** (1951), 302–314; MR **13**, 324], Kawada's definition being essentially the restriction of the author's to the 1-cohomology group.

Results: All these kinds of differents, for all n, are equal to the usual different of algebraic number theory defined by the trace from L to K; from the author's definition it is proved that the different is not zero, and that a prime $\mathfrak{P}$ divides the different if and only if it is ramified in L/K. *G. Whaples* (Bloomington, Ind.)

6836:

Kuniyoshi, Hideo. On the cohomology groups of $\mathfrak{p}$-adic number fields. Proc. Japan Acad. **34** (1958), 609–611.

Let K be a complete field under a discrete valuation and let L be a finite separable extension field of K with separable residue class field. Let R and Λ be the rings of integral elements of K and L. The author computes explicitly the homology and cohomology groups of any two-sided Λ-module A, considering Λ as an algebra over R. In the special case when $\lambda a = a\lambda$ for all $a \in A$, $\lambda \in \Lambda$, the result is as follows. Let $L = R + R\vartheta + \cdots + R\vartheta^{n-1}$ (such a ϑ always exists) and let ϑ satisfy $f(x) = 0$ over R. Then $H^{2r+1}(\Lambda, A) \cong H_{2r+2}(\Lambda, A) \cong A_{f'(\vartheta)}$ and $H^{2r+2}(\Lambda, A) \cong H_{2r+1}(\Lambda, A) \cong A/f'(\vartheta) \cdot A$, for $r \geqq 0$, where $A_{f'(\vartheta)} = \{a \in A \mid f'(\vartheta)a = 0\}$. The homology and cohomology groups are of period 2 also in the general case. The author briefly indicates a proof that this is true also when R and Λ are rings of integral elements of algebraic number fields.

G. Whaples (Bloomington, Ind.)

6837:

Kuniyoshi, Hideo. Cohomology groups of maximal orders of $\mathfrak{p}$-adic simple algebras. Duke Math. J. **27** (1960), 387–396.

Let k be a $\mathfrak{p}$-adic number field, R the ring of all $\mathfrak{p}$-integers of k, and $\mathfrak{A}$ a central simple algebra over k. Let Λ be a maximal order in A, considered as an algebra over R, and M a two sided Λ-module. The author proves that for all $r \geqq 1$, $H^{r+2}(\Lambda, M) \cong H^r(\Lambda, M)$ and $H_{r+2}(\Lambda, M) \cong H_r(\Lambda, M)$. He in fact derives explicit formulas for these cohomology and homology groups. He concludes by briefly indicating a proof that the homology and cohomology groups have period 2 also in case of modules over a maximal order of a simple algebra whose center is an algebraic number field.

On p. 395 the misprint of $\Lambda_{\mathfrak{R}}$ for Λ_R occurs in line 4 and in the right sides of isomorphisms (30) and (31).

G. Whaples (Bloomington, Ind.)

6838:

Yamasaki, Hisashi. On products of Hochschild groups. III. Tôhoku Math. J. (2) **11** (1959), 147–161.

Exercices variés sur des thèmes connus. Il s'agit de la cohomologie de Hochschild, aussi bien dans le cas "relatif" que dans le cas "absolu". *H. Cartan* (Paris)

6839:

Matlis, Eben. Divisible modules. Proc. Amer. Math. Soc. **11** (1960), 385–391.

R désigne un anneau intègre, Q son corps des fractions. L'auteur donne une série de résultats se rapportant au problème de savoir quand un R-module divisible D est quotient d'un module injectif, ce qui implique (th. 1.1) que le sous-module de torsion D_T est facteur direct dans D. Voici quelques échantillons de ces résultats.

Si tout R-module divisible est quotient d'un injectif, et si $R \neq Q$, la dimension homologique $\text{hd}_R\, Q$ est égale à 1. Réciproquement, si $\text{hd}_R\, Q = 1$, tout R-module A possède un plus grand sous-module H qui soit quotient d'un injectif, et A/H est "h-réduit" (i.e.: si un sous-module de A/H est quotient d'un injectif, il est nul).

Dans chacun des deux cas suivants, tout module divisible est quotient d'un module injectif: (1) Q est un R-module de type dénombrable; (2) R est noethérien, et de dimension de Krull égale à 1 (i.e.: tout idéal premier $\neq 0$ est maximal). *H. Cartan* (Paris)

6840:

Heller, Alex. The loop-space functor in homological algebra. Trans. Amer. Math. Soc. **96** (1960), 382–394.

The paper is a sequel to Ann. of Math. (2) **68** (1958), 484–525 [MR **20** #7051]. A subset $\mathfrak{S}$ of the maps of an additive category $\mathscr{X}$ is said to be an ideal if (1) for each $A, B \in \mathscr{X}$ the intersection $\mathfrak{S} \cap \text{Hom}\,(A, B)$ is a subgroup of $\text{Hom}\,(A, B)$, and (2) for $f: A \to B$, $g: B \to C$ we have $gf \in \mathfrak{S}$ whenever $f \in \mathfrak{S}$ or $g \in \mathfrak{S}$. An object in $\mathscr{X}$ is said to be in $\mathfrak{S}$ when its identity map is in $\mathfrak{S}$. The quotient category $\mathscr{X}/\mathfrak{S}$ is defined, and particularly the case is studied when the ideal $\mathfrak{S}$ is generated by identity maps. For an abelian category $\mathscr{X}$ with enough projectives and for the ideal $\mathfrak{P}[\mathscr{X}]$ generated by the projectives in $\mathscr{X}$ a functor $\Omega: \mathscr{X}/\mathfrak{P}[\mathscr{X}] \to \mathscr{X}/\mathfrak{P}[\mathscr{X}]$, called the loop-space functor of $\mathscr{X}$, is defined, in analogy to Eckmann and Hilton's case of category of modules, and in duality to the suspension functor. Further, with $\mathscr{X}^s$ denoting the category of proper s.e.s. (=short exact sequences) in an abelian category $\mathscr{X}$, $\mathscr{X}^s$ has enough projectives when $\mathscr{X}$ does, and a functor $\Gamma: \mathscr{X}^s/\mathfrak{P}[\mathscr{X}^s] \to \mathscr{X}^s/\mathfrak{P}[\mathscr{X}^s]$, the permutation functor of $\mathscr{X}^s$, is introduced; from this, and from Ω, "generalized commutation equivalences" (in particular, "canonical commutation equivalences" and "canonical permutation equivalences") of certain composite functors $\mathscr{X}^s/\mathfrak{P}[\mathscr{X}^s] \to \mathscr{X}/\mathfrak{P}[\mathscr{X}]$ are derived. Γ^3 and Ω are naturally equivalent, and it follows that to each pair $A \in \mathscr{X}^s$, $B \in \mathscr{X}$, is associated a certain exact sequence of homomorphism groups in $\mathscr{X}/\mathfrak{P}[\mathscr{X}]$. Finally the paper introduces a functor on the category Ext $\mathscr{X}$ with values in a certain quotient category $\mathcal{E}\mathscr{X}$, $\mathscr{X}$ being abelian with enough projectives, and the kernel of this functor measures the extent to which projectives fail to be injective. *T. Nakayama* (Nagoya)

6841:

Hilton, P. J.; Ledermann, W. Homology and ringoids. III. Proc. Cambridge Philos. Soc. **56** (1960), 1–12.

[Parts I, II: same Proc. **54** (1958), 152–167; **55** (1959), 149–164; MR **20** #7050a; **21** #3475.] The notions of direct sum and direct product (finite or infinite) in homological ringoids are defined and their basic properties are established. *S. Eilenberg* (New York)

GROUPS AND GENERALIZATIONS

See also 6724, 7031.

6842:

Rapaport, Elvira Strasser. On the commutator subgroup of a knot group. Ann. of Math. (2) **71** (1960), 157–162.

This paper considers a finitely generated group G that has the following two properties: (i) the commutator quotient group G/G' is infinite cyclic; (ii) it has at least one presentation in which the number of generators exceeds the number of relations. (Every knot group has properties (i) and (ii).) Any group G having properties (i) and (ii) has an Alexander polynomial $\Delta(t) = c_0 + c_1 t + \cdots + c_d t^d$ such that $\Delta(1) = 1$. The principal result is that if $c_0 c_d = \pm 1$ then G'/G'' is free abelian of rank d, but that if $c_0 c_d \neq \pm 1$ then G'/G'' is not finitely generated although each of its finitely generated subgroups is of rank $\leqq d\,(\text{rank}\, G - 1)$. Simple examples show that G' itself need not be finitely generated even if $c_0 c_d = \pm 1$. Furthermore, any polynomial $P(t) = c_0 + c_1 t + \cdots + c_d t^d$ satisfying $P(1) = 1$ and $c_0 c_d = \pm 1$ is the Alexander polynomial of some group G having properties (i) and (ii). It is also shown that the second homology group of any group G having properties (i) and (ii) is trivial; this was previously known only for knot groups G. *R. H. Fox* (Princeton, N.J.)

6843:

Curzio, Mario. Su alcuni gruppi complementati. Ricerche Mat. **8** (1959), 172–179.

Results of Zacher [Rend. Accad. Sci. Fis. Mat. Napoli (4) **19** (1952), 200–206; Rend. Sem. Mat. Univ. Padova **22** (1953), 113–122; MR **15** 775, 286] on finite soluble groups with relatively complemented or complemented subgroup lattices are extended to a restricted class of infinite groups. It is shown that if the group G is countable, soluble, and has elements of only finitely many prime orders $p_1 > p_2 > \cdots > p_n$, and if the subgroup lattice of G is relatively complemented, then G is periodic, the relation of normality is transitive for subgroups of G, the subgroups of G are elementary abelian, and G is (consequently) "dispersible", that is to say, it has a normal Sylow p_1-subgroup S_1, and modulo S_1 a normal Sylow p_2-subgroup S_2, and so on. It is also shown that the groups in which every subgroup has a unique complement are the locally cyclic groups whose elements have finite square-free orders. *B. H. Neumann* (Manchester)

6844:

Wright, C. R. B. On groups of exponent four with generators of order two. Pacific J. Math. **10** (1960), 1097–1105.

Denote by $G(n)$ the group, generated by n elements of order 2, in which the fourth power of every element is 1.

The author shows that $G(n)$ is nilpotent of class at most $n+1$, and conjectures that the nilpotency class of $G(n)$ is precisely $n+1$ for $n>2$. *L. J. Paige* (Los Angeles, Calif.)

6845:

Higman, Graham. Lie ring methods in the theory of finite nilpotent groups. Proc. Internat. Congress Math. 1958, pp. 307–312. Cambridge Univ. Press, New York, 1960.

Let G be a group which has a normal series $G=H_1\supset H_2\supset\cdots\supset H_i\supset\cdots$ such that the commutator subgroup $[H_i, H_j]\subset H_{i+j}$ and such that $\bigcap_{i=1}^{\infty} H_i=1$. A Lie ring L may be associated with G as follows: the underlying additive group of L is the direct sum of the factor groups H_i/H_{i+1}, while multiplication in L is defined so that $g_iH_{i+1}\cdot g_jH_{j+1}=[g_i, g_j]H_{i+j+1}$. This well-written expository paper discusses relations between the structure of L and that of G in two situations: (1) G is nilpotent and has a finite regular group of automorphisms; (2) G satisfies an identical relation $x^n=1$. Some unsolved problems are pointed out. *G. Leger* (Cleveland, Ohio)

6846:

Hall, Marshall, Jr. Current studies on permutation groups. Proc. Sympos. Pure Math., Vol. 1, pp. 29–41. American Mathematical Society, Providence, R.I., 1959.

This article is a critique of current work. Other (extensive) surveys are available: W. A. Manning, *Primitive groups* [Stanford Univ. Press, Stanford, Calif., 1921, out of print], H. Wielandt, *Permutationsgruppen* [lecture notes, Tübingen, 1955]. A permutation group is an abstract object consisting of a group and a certain set of subgroups thereof. Two chief announcements are made of new results. (1) E. Parker has proved that if $n=kp+r$, with k, r integers and p prime, and if $k<p^2$, $k<r(r-1)/2$, $r\geqq 12$, then a group of degree n cannot be more than r-ply transitive. (2) Let G have degree n; let the subgroup H fixing $n-m$ letters be transitive on the remaining m letters. If G is w-ply transitive, it is known that $w\geqq 2$; if H is primitive, $w\geqq n-m+1$. If $2<w<n-m+1$, the structure of G turns out to be related to a problem in finite projective geometry. *J. L. Brenner* (Madison, Wis.)

6847:

★Convegno Internazionale di Teoria dei Gruppi Finiti e Applicazioni. Promosso dalla Università di Firenze, Firenze, 11–13 aprile 1960. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1960. vii+157 pp. L. 2500.

Contains 9 addresses, and 10 summaries of papers which are to appear elsewhere. The addresses will be reviewed separately.

6848:

Wielandt, H. Entwicklungslinien in der Strukturtheorie der endlichen Gruppen. Proc. Internat. Congress Math. 1958, pp. 268–278. Cambridge Univ. Press, New York, 1960.

A survey of recent progress in finite group theory, with particular reference to relations between arithmetic and normal structure. *Graham Higman* (Chicago, Ill.)

6849:

Curzio, Mario. Sui gruppi risolubili a fattoriali supersolubili. Ricerche Mat. **9** (1960), 82–90.

The author determines those finite soluble groups all of whose proper factor groups are supersoluble, and those finite supersoluble groups which have a unique composition series; related results are also presented.

Graham Higman (Chicago, Ill.)

6850:

Coxeter, H. S. M. Symmetrical definitions for the binary polyhedral groups. Proc. Sympos. Pure Math., Vol. 1, pp. 64–87. American Mathematical Society, Providence, R.I., 1959.

Die binären Polyedergruppen können auf vier Wegen eingeführt werden: (a) durch die definierenden Relationen (*) $R^l=S^m=T^n$ mit passendem l, m und n; (b) als endliche Quaternionengruppen; (c) als endliche Gruppen von Clifford Verschiebungen in einem sphärischen Raum; (d) als Fundamentalgruppen gewisser dreidimensionaler Mannigfaltigkeiten. Vorliegende Arbeit gibt einen klaren Ueberblick über den Zusammenhang obiger Definitionen. Insbesondere wird gezeigt, wie eine der Erzeugenden in (*) weggelassen werden kann und man dadurch symmetrische Relationen in zwei Erzeugenden erhält.

J. J. Burckhardt (Zürich)

6851:

Nagata, Masayoshi. On the fourteenth problem of Hilbert. Proc. Internat. Congress Math. 1958, pp. 459–462. Cambridge Univ. Press, New York, 1960.

The 14th problem of Hilbert is here formulated as follows: "Let G be a subgroup of the full linear group of the polynomial ring in indeterminates $x_1, x_2, \cdots, x_n$ over a field k, and let $\mathfrak{O}$ be the set of elements of $k[x_1, x_2, \cdots, x_n]$ which are invariant under G. Is $\mathfrak{O}$ finitely generated?" Although this problem has an affirmative answer in special cases, an example is given to show that as it stands, this question must be answered negatively.

H. T. Muhly (Iowa City, Iowa)

6852:

Jakubík, Ján. Direkte Zerlegungen der teilweise geordneten Gruppen. Czechoslovak Math. J. **10 (85)** (1960), 231–243. (Russian. German summary)

Let G be a partially ordered group. Define $K=\{x\in G \mid -y\leqq x\leqq y, \text{some } y\in G^+\}$. Then K is a directed, normal subgroup of G which is, roughly speaking, the part of G in which the order is interesting. It is known that direct product decompositions of directed partially ordered groups have a common refinement [see E. P. Šimbireva, Mat. Sb. (N.S.) **20 (62)** (1947), 145–178; MR **8**, 563]. In this paper it is shown that every two direct decompositions of the (not necessarily directed) group G have a common refinement if and only if this is so in G/K and, for any two direct factors A, B of G, $(A+K)\cap(B+K)=(A\cap B)+K$. The paper also contains some results on decompositions of directed groups. For example, if a directed group G is expressed as a cartesian product of two subsets X and Y, then X and Y are actually subgroups and G is the direct product of these subgroups.

R. S. Pierce (Seattle, Wash.)

6853:

Jaffard, Paul. Sur le spectre d'un groupe réticulé et l'unicité des réalisations irréductibles. Ann. Univ. Lyon. Sect. A (3) **22** (1959), 43–47.

If G is an abelian lattice-ordered group, then Lorenzen showed [Math. Z. **45** (1939), 533–553; MR **1**, 101] that there exists an l-isomorphism σ of G onto a subdirect product of a cardinal product $\prod G_\gamma$ $(\gamma \in \Gamma)$ of linearly ordered groups G_γ. For each $\alpha \in \Gamma$ let σ_α be the natural homomorphism of $\prod G_\gamma$ $(\gamma \in \Gamma)$ onto $\prod G_\gamma$ (all $\gamma \neq \alpha$). If for each $\alpha \in \Gamma$ the product $\sigma\sigma_\alpha$ is not one to one, then this representation is said to be irreducible. The author showed [J. Math. Pures Appl. (9) **32** (1953), 203–280; MR **15**, 284] that any two irreducible representations of G are essentially the same. In this paper he gives a short elegant topological proof of the uniqueness of an irreducible representation of G. The set S of all prime l-ideals of G is topologized in a natural way so that it is a T_0 space. A subset F of S is called irreducible if it is dense in S and if for each $x \in F$, $F \setminus \{x\}$ is not dense in S. The irreducible subsets of S correspond to the irreducible representations of G, and it is shown that a T_0 space admits at most one irreducible subset. *P. F. Conrad* (New Orleans, La.)

6854:

Conrad, Paul. The structure of a lattice-ordered group with a finite number of disjoint elements. Michigan Math. J. 7 (1960), 171–180.

$A_1, \cdots, A_n$ étant des groupes totalement ordonnés (non nécéssairement abéliens), l'auteur appelle somme lexicographique des A_i les divers groupes réticulés obtenus à partir des A_i par une succession de sommes directes ordonnées et d'extensions lexicographiques par des groupes totalement ordonnés quelconques. Ceci posé, l'auteur démontre que si L est un groupe réticulé possédant n éléments disjoints $a_1, \cdots, a_n$ (c'est-à-dire tels que $0<a_i$ $(1 \leqq i \leqq n)$ et $a_i \cap a_j = 0$ si $i \neq j$) et n'en possédant pas $n+1$, le groupe L est somme lexicographique des n groupes totalement ordonnés $A_1, \cdots, A_n$ définis ainsi: A_i est le sous-groupe de L engendré par les $x \in L$ tels que $x \cap a_j = 0$ pour tout $j \neq i$. Le rapporteur remarque que dans le cas où L est un groupe abélien, ceci est une conséquense immédiate de ses propres résultats concernant les groupes n'ayant qu'un nombre fini de filets [Publ. Sci. Univ. Alger Sér. A **1** (1954), 197–222; MR **17**, 346]. *P. Jaffard* (Paris)

6855:

Zak, J. Method to obtain the character tables of non-symmorphic space groups. J. Mathematical Phys. **1** (1960), 165–171.

Die Charaktere der irreduziblen Darstellungen derjenigen Raumgruppen, die einen Punkt invariant lassen, sind bekannt. Um diejenigen der übrigen zu erhalten, zerlegt Verfasser die Gruppen nach einer Untergruppe vom Index 2 oder 3 und setzt die Charaktere der ganzen Gruppe nach bekannten Methoden aus denjenigen der Untergruppen zusammen. Als Biespiele werden O_h^2 und T^4 behandelt. *J. J. Burckhardt* (Zürich)

6856:

Brameret, Marie-Paule. Sur certains complexes et éléments remarquables d'un demi-groupe. C. R. Acad. Sci. Paris **250** (1960), 1417–1418.

Deux complexes H et K d'un demi-groupe $\mathcal{D}$ sont coréflectifs si $ab \in H \Leftrightarrow ba \in K$. Un élément $a \in \mathcal{D}$ est réflectible à gauche si $ax = ay \Rightarrow xa = ya$. Un élément $a \in \mathcal{D}$ est commutatible à droite si $xay = x'ay' \Rightarrow ayx = ay'x'$.

Le but de la note est l'étude des ces notions. Par example: un demi-groupe simple à droite dont tous les éléments sont commutatibles à droite est un groupe abélien. Les autres propositions sont d'un caractère analogue. *Š. Schwarz* (Bratislava)

6857:

Clifford, A. H. Basic representations of completely simple semigroups. Amer. J. Math. **82** (1960), 430–434.

This note was written to complete the results of a paper in Amer. J. Math. **64** (1942), 327–342 [MR **4**, 4]. It was shown by D. Rees [Proc. Cambridge Philos. Soc. **36** (1940), 387–400; MR **2**, 127] that a semigroup S is completely simple if and only if it is a regular matrix semigroup over a group G with zero. Clifford showed in the earlier paper that every representation $\mathfrak{T}^*$ of S induces a representation $\mathfrak{T}$ of the group G (or $\mathfrak{T}^*$ may be called an "extension" of $\mathfrak{T}$). Not every representation $\mathfrak{T}$ of G is extendible to a representation of S. If $\mathfrak{T}$ is extendible then the extension $\mathfrak{T}_0^*$ of least degree is uniquely determined. The mapping $\mathfrak{T} \to \mathfrak{T}_0^*$ is one-to-one. If $\mathfrak{T}$ is irreducible so is $\mathfrak{T}_0^*$. This note proves the converse, and hence all irreducible representations of the completely simple semigroup S are obtained as the basic extensions $\mathfrak{T}_0^*$ of the extendible irreducible representations $\mathfrak{T}$ of the group G. *H. H. Campaigne* (Jessup, Md.)

6858:

Gluskin, L. M. Densely imbedded ideals of semigroups. Dokl. Akad. Nauk SSSR **131** (1960), 1004–1006 (Russian); translated as Soviet Math. Dokl. **1**, 361–364.

An ideal A of a semigroup S is said to be densely imbedded in S if: (1) every nonisomorphic homomorphism of S induces a nontrivial homomorphism of A. Many results about densely imbedded ideals are announced. Typical result: Let A be a semigroup such that $a, a' \in A$ and $ax = a'x$ and $xa = xa'$ for all $x \in A$ imply $a = a'$. Let S be a semigroup containing A as a densely imbedded ideal. Then a semigroup S' is isomorphic with S if and only if it contains a densely imbedded ideal isomorphic to A. *E. Hewitt* (Seattle, Wash.)

6859:

Norton, Donald A. A note on associativity. Pacific J. Math. **10** (1960), 591–595.

Un groupoide $G = E(\cdot)$ est tri-associatif si pour trois éléments distincts quelconques x, y, z de E on a (1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$; si x, y, z ne sont pas distincts, la relation (1) peut avoir lieu ou non. G est di-associatif si (1) est vraie toutes les fois que deux des éléments x, y, z sont égaux. Enfin G est mono-associatif si $x \cdot (x \cdot x) = (x \cdot x) \cdot x$ quel que soit x dans E. Alors: (i) tout quasigroupe tri-associatif a un élément neutre bilatère; (ii) tout quasigroupe tri-associatif Q construit sur un ensemble de 17 éléments au moins et dont la diagonale contient également 17 éléments au moins est di-associatif; (iii) si un quasigroupe $Q = E(\cdot)$ satisfait pour tous x, y dans E à l'identité

$x\cdot(x\cdot y)=(x\cdot x)\cdot y$, alors Q est mono-associatif. On conjecture que le nombre 17 n'est pas le minimum pour le théorème (ii). *A. Sade* (Marseille)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 6800, 6845, 7129.

6860:

Glicksberg, Irving. Some special transformation groups. Proc. Amer. Math. Soc. **11** (1960), 315–318.

Two of the main results are the following. Theorem 2: Let H be a compact connected abelian group, and let G be an equicontinuous group of self-homeomorphisms of H containing all translations. Then $g\in G$ is of the form $g(h)=h_g\sigma_g(h)$, where σ_g is an automorphism of H. If H is also finite-dimensional there is an integer k for which $\sigma_g{}^k(h)=h$ for all g and h. Theorem 3: Let H be a finite-dimensional compact connected abelian group, G an equicontinuous group of automorphisms of H. Then for some integer k, every element of G is of period k.

D. Montgomery (Princeton, N.J.)

6861:

Glicksberg, Irving. A remark on some almost periodic compactifications. Michigan Math. J. **7** (1960), 133–135.

Let G be a noncompact, locally compact abelian group whose character group is not totally disconnected, and G^* be the almost periodic compactification of G. The author shows that the complement of G in G^* has G^* as its Stone-Čech compactification. *K. deLeeuw* (Princeton, N.J.)

6862:

Čarin, V. S. On locally compact soluble groups satisfying the minimal condition for closed subgroups. Dokl. Akad. Nauk SSSR **131** (1960), 1036–1037 (Russian); translated as Soviet Math. Dokl. **1**, 392–393.

The author proves that for locally compact groups the descending chain condition for closed subgroups is equivalent to that for closed abelian subgroups, i.e., every descending chain of closed subgroups of a closed abelian subgroup breaks off. This is a topological analogue to a result on abstract groups due to Černikov [Mat. Sb. (N.S.) **28 (70)** (1951), 119–129; MR **12**, 477].

K. A. Hirsch (St. Louis, Mo.)

6863:

Macbeath, A. M.; Swierczkowski, S. Limits of lattices in a compactly generated group. Canad. J. Math. **12** (1960), 427–437.

The first result is a useful lemma on fundamental domains. It is observed that, if f is continuous, $\geqq 0$ and has compact support in a locally compact space with a positive measure μ, and if one writes $X_\tau=\{x\mid f(x)>\tau\}$, then $\mu(X_\tau)$, for $\tau\geqq 0$, is decreasing, hence continuous except at a countable set T of values of τ, so that, for τ not in T, the boundary $\bar{X}_\tau-X_\tau$ of X_τ has the measure 0. This is applied to the construction of a fundamental domain F for a discrete subgroup H of a locally compact group G, countable at infinity, such that F is a countable union of locally closed sets and that the boundary of F is of measure 0; if G/H is compact, F can be chosen relatively compact. The main theorem concerns convergent sequences of discrete subgroups in the following sense [cf. C. Chabauty, Bull. Soc. Math. France **78** (1950), 143–151; MR **12**, 479]. If H and all H_n are discrete subgroups of G, the sequence H_n is said to tend to H if, for every compact subset K of G and every neighborhood V of e, $H_n\cap K\subset VH$ and $H\cap K\subset VH_n$ for n large enough; the H_n are called uniformly discrete if there is a neighborhood V of e such that $H_n\cap V=\{e\}$ for all n. Now assume that G/H is compact (then the Haar measure μ on G is bi-invariant). If the H_n are uniformly discrete and tend to H, and if G is compactly generated, it is shown that H is finitely generated, that G/H_n is compact for large n (and uniformly so, i.e., there is a compact set K such that $G=KH_n$ for large n), and that $\mu(G/H_n)$ tends to $\mu(G/H)$; the question whether H_n, for large n, must then be isomorphic to H is left open. Some related results are also given (including such for groups which are not compactly generated), as well as examples to show that the conditions in the main theorem are all necessary (in particular, it would not be enough to assume that G/H has finite measure).

A. Weil (Princeton, N.J.)

6864:

Yokota, Ichiro. Embeddings of projective spaces into elliptic projective Lie groups. Proc. Japan Acad. **35** (1959), 281–283.

By means of an explicit formula the octavian plane is embedded into the elliptic group (F_4).—The existence of such an embedding is classical, as this elliptic plane is a symmetric manifold in the Cartan sense (the geodesic reflexion in an automorphism). The embedding can be geometrically constructed by mapping a point A of the plane upon the harmonic perspectivity $\pi_{A,A}^{-1}$ with the centre A and the axis A [see H. Freudenthal, Nederl. Akad. Wetensch., Proc. Ser. A. **58** (1955), 277–285; MR **16**, 900; especially formula 11.3.3]. This is the most natural embedding. The author's embedding seems to be rather arbitrary. Its geometrical background is not clear.

H. Freudenthal (New Haven, Conn.)

6865:

Rodrigues, A. A. Martins. Equivalent sub-manifolds of Lie groups. An. Acad. Brasil. Ci. **32** (1960), 191–192.

Let G be a Lie group, G^* the Lie algebra of G and Ω the canonical left-invariant differential form of G with values in G^*. Let S_1 and S_2 be two connected submanifolds of G, and Ω_1 and Ω_2 the restriction of Ω to S_1 and S_2, respectively. The author shows that there exists an element g of G such that $g(S_1)=S_2$ if and only if there exists a diffeomorphism $\phi:S_1\to S_2$ with $\phi^*\Omega_2=\Omega_1$.

S. Kobayashi (Vancouver, B.C.)

6866:

Kuranishi, Masatake. On the local theory of continuous infinite pseudo groups. I. Nagoya Math. J. **15** (1959), 225–260.

The author develops the theory of continuous infinite pseudo-groups of E. Cartan by introducing parameter local groups. Chapter I is devoted to formal analytic mappings which generalize formal power series. An (F)-vector space H is, by definition, a direct product of finite-dimensional vector spaces B^i, $i=0, 1, 2, \cdots$, with the

property that there exist integers p, k and a real number m_1 such that

$$m_1(i-k) \leq \dim (B^0 + \cdots + B^{i-1}) \leq m_1(i+k)^p$$

for sufficiently large i. Every element ξ of H can be uniquely written as $\xi = \xi^0 + \xi^1 + \xi^2 + \cdots$, where $\xi^i \in B^i$; it is not assumed that almost all ξ^i are zero. An element of B^i is called a homogeneous element of degree i. A basis $h_1, h_2, h_3, \cdots$ of H is constructed by first choosing a basis in each of the B^i's and then introducing a lexicographical order in the set of these elements. A formal analytic map F of H into another (F)-vector space H' is defined by

$$H \ni \xi \to \sum_i (\sum_m F_m{}^i(\xi))h_i',$$

where $h_1', h_2', \cdots$ is a basis of H' and each $F_m{}^i$ is a homogeneous polynomial function on H of degree m and weight $\leq \deg h_i + km$. The integer k is called the degree of F. In Chapter II, using formal analytic mapping, the author defines formal Lie (F)-groups and formal Lie (F)-algebras. A formal Lie (F)-group with parameter space H is a formal analytic mapping of $H+H$ into H satisfying certain conditions; a similar definition holds for a formal Lie (F)-algebra. He proves Lie's fundamental theorems by establishing a one-to-one correspondence between the formal Lie (F)-groups and the formal Lie (F)-algebras.

S. Kobayashi (Vancouver, B.C.)

6867:

Graev, M. I. Irreducible unitary representations of certain classes of real simple Lie groups. Dokl. Akad. Nauk SSSR **127** (1959), 13–16. (Russian)

Continuing his work on the irreducible unitary representations of the real simple Lie groups, the author gives simple explicit formulas for the (generalized) traces of the so-called analytic series of representations of certain of the automorphism groups of bounded complex domains. Specifically, he treats the groups of complex matrices of order $2p$ (p integral) and determinant 1 leaving invariant the form $\sum_{j=1}^{p} (x_j\bar{x}_j - x_{j+p}\bar{x}_{j+p})$ and also either the quadratic form $\sum_j x_j x_{j+p}$ (the series 'A_p') or else the bilinear form $\sum_j (x_j y_{j+p} - x_{j+p} y_j)$.

I. E. Segal (Cambridge, Mass.)

6868:

Curtis, Charles W. On the dimensions of the irreducible modules of Lie algebras of classical type. Trans. Amer. Math. Soc. **96** (1960), 135–142.

In an earlier paper [J. Math. Mech. **9** (1960), 307–326; MR **22** #1634], the author has developed the irreducible restricted representation theory for Lie algebras with non-degenerate Killing forms over algebraically closed fields of prime characteristic. In this paper, he selects a subclass of these irreducible representations such that the representation-space in question is obtained by reduction of coefficients modulo p and field extension from one of the same dimension for an analogous complex semi-simple Lie algebra. Thus it is possible to translate Weyl's formula for the dimension in the complex case to the representation-spaces in question. The conditions imposed on the representation by the author seem to be strongly restrictive, but he checks the reader's tendency to chafe at them by demonstrating a simple example where Weyl's formula does not apply, at least in its most direct form.

G. B. Seligman (New Haven, Conn.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

6869:

Mostert, Paul S.; Shields, Allen L. One-parameter semigroups in a semigroup. Trans. Math. Soc. **96** (1960), 510–517.

Let S be a Hausdorff topological semigroup with identity 1. Denote by $H(1)$ the set of elements with two-sided inverses with respect to 1. A one-parameter semigroup in S is a continuous one-to-one function $\sigma:[0, 1] \to S$ such that $\sigma(0)=1$ and $\sigma(a+b)=\sigma(a)\cdot\sigma(b)$ for all $a, b \in [0, 1]$ for which $a+b \in [0, 1]$. The following result on the existence of one-parameter semigroups is established. Theorem: Let S be compact and assume that $H(1)$ is not open in S. Let V be a neighborhood of 1 containing no other idempotents. Then S contains a one-parameter semigroup σ such that $\sigma(a) \notin H(1)$ for $0<a\leq 1$; moreover, $\sigma(a)=\sigma(b)\cdot g$, $g \in H(1)$, implies $a=b$ and $g=1$. This is an extension of a previous result of the authors [Ann. of Math. (2) **65** (1957), 117–143; MR **18**, 809] in which it was assumed that $H(1)$ is a Lie group. The present result has already found numerous applications in semigroups, and has been used by various authors in proving, e.g., that a one-dimensional compact connected semigroup with zero and unit is arc-wise connected, and that a compact connected semigroup with unit contains a generalized arc [Koch and Wallace, Trans. Amer. Math. Soc. **88** (1958), 277–287; MR **20** #1729; Hunter, ibid. **93** (1959), 356–368; MR **22** #82; Koch, Pacific J. Math. **9** (1959), 723–728; MR **21** #7269]. The authors show further that if S is a compact semigroup with zero and identity and no other idempotents, and $H(1)$ is not open, then S contains a standard thread, and S is arcwise connected. A result on the local imbeddability of locally compact semigroups in Lie groups is stated. The paper is concluded with some examples and conjectures.

R. J. Koch (Baton Rouge, La.)

6870:

Schwarz, Štefan. A theorem on normal semigroups. Czechoslovak Math. J. **10 (85)** (1960), 197–200. (Russian summary)

If S is a compact semigroup and $a \in S$, then $\Gamma(a)$, the closure of $\{a, a^2, a^3, \cdots\}$, is known to contain a unique idempotent. If e is the idempotent in $\Gamma(a)$ we say a belongs to e. It is known that if S is a compact commutative semigroup and if a belongs to e_1 and b belongs to e_2, then ab belongs to e_1e_2. The theorem proved in this paper extends this result to a wider class of semigroups which includes compact normal semigroups. (A semigroup S is said to be normal if and only if $xS=Sx$ for each x in S.) The class of semigroups considered are those compact semigroups, S, such that each idempotent of S is contained in the center of S.

Anne Lester (New Orleans, La.)

6871:

Rothman, Neal J. Embedding of topological semigroups. Math. Ann. **139**, 197–203 (1960).

It is well known that a commutative semigroup with cancellation can be embedded in a group. [See B. Gelbaum, G. K. Kalisch and J. M. H. Olmstead, Proc. Amer. Math. Soc. **2** (1951), 807–821; MR **13**, 206.] If P is the isomorphism of such a topological semigroup S into G, the

group generated by S, then S is said to be embeddable in G if and only if G is a topological group and P is a homeomorphism of S onto $P(S)$, with the relative topology induced by G. The main theorem of the present paper is as follows. If S is a commutative semigroup with cancellation, then a necessary and sufficient condition that S be embeddable as an open set of G is that S has the following property (F): If x and y are in S and V is open containing x, then there is an open set W with $y \in W$ such that $xy \in \bigcap[Vy' : y' \in W]$ and $yx \in \bigcap[y'V : y' \in W]$. There are other results pertaining to the embedding of S in G, where S is assumed to be a commutative subsemigroup with cancellation in a compact semigroup T such that if $x \in S$ and $y \in T$ and $xt \in S$ then $t \in S$. In particular, it is shown that if S has property F, or the property that if $x, y \in S$ and $x \neq y$, then $x \in Sy$ or $y \in Sx$, then S is embeddable in G. The final section of the paper is devoted to deriving sufficient conditions for a homomorphism f of a semigroup S onto a semigroup T to be an open map.

{In a communication with the reviewer the author states that in Lemma 1.1 and Corollary 1.1.1, S is assumed to be a semigroup with cancellation, for otherwise there are examples where the conclusions do not hold.}

Anne Lester (New Orleans, La.)

FUNCTIONS OF REAL VARIABLES

See also 6973.

6872:

Strodt, Walter. Remarks on the Euler-Maclaurin and Boole summation formulas. Amer. Math. Monthly **67** (1960), 452–454.

Soit f continue dans $[x, x+1]$,

$$If(x) = \int_0^1 f(x+t)\,dt, \quad Af(x) = \tfrac{1}{2}[f(x)+f(x+1)].$$

L'auteur constate une complète similarité entre la formule de Taylor

$$(*) \qquad f(x) = \sum_0^n \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n$$

et les formules de sommation d'Euler-Maclaurin et de Boole, lesquelles découlent de (*) en y remplaçant $f^{(k)}(a)$ par $If^{(k)}(a)$ [resp. $Af^{(k)}(a)$] et $(x-a)^k$ par $B_k(x-a)$ (les polynômes de Bernoulli) [resp. par $E_k(x-a)$ (les polynômes d'Euler)]. De plus, le reste dans les formules mentionnées résulte aussi par l'intégration par parties répetée, ainsi que dans (*).

M. Tomić (Belgrade)

6873:

Iosifescu, Marius. Über die differenzialen Eigenschaften der reellen Funktionen einer reellen Veränderlichen. Rev. Math. Pures Appl. **4** (1959), 457–466.

Cet article contient les démonstrations de résultats annoncés dans une note antérieure [C. R. Acad. Sci. Paris **248** (1959), 1918–1919; MR **21** #4999]. En voici les idées directrices: (1) Les nombres rationnels sur (a, b) ayant été rangés en une suite $r_1, r_2, \cdots, r_j, \cdots$, la fonction auxiliaire f_j^+ est définie pour $r_j \leq x < b$ par $f_j^+(x) = \sup f(y)$ pour $y \in [r_j, x]$. Ses associées f_+^j, f_j^- et f_-^j sont définies similairement. N_j' désigne l'ensemble des x de (a, b) où les fonctions auxiliaires de rang j n'admettent pas simultanément une dérivée finie, N_f' l'union des N_j', E_f l'ensemble des x où f présente un maximum ou un minimum strict. Il est montré de manière élémentaire qu'en tout point x_0 n'appartenant ni à E_f ni à N_f' et tel que, sur un intervalle (x_0-h, x_0), $h>0$, $f(x)$ soit $<f(x_0)$, le nombre dérivé supérieur droit $D^+f(x_0)$ est égal au nombre dérivé inférieur gauche $D_-f(x_0)$. (2) Si f possède la propriété de Darboux, à tout x_0 isolé à gauche correspond un $h>0$ tel que $f(x)$ soit constamment positif ou constamment négatif pour $x \in (x_0-h, x_0)$. (3) Si φ est une fonction monotone bornée définie sur (a, b), l'image par φ de l'ensemble des points x où n'existe pas une dérivée finie de φ est de mesure lebesguienne nulle. {Remarque du rapporteur. Des fonctions auxiliaires extrémales sont utilisées dans un but analogue par O. Haupt et Chr. Pauc, S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 51–55 [MR **11**, 337].}

Chr. Y. Pauc (Nantes)

6874:

Iosifescu, Marius. Über eine Erweiterung eines Satzes von S. Stoilow. Rev. Math. Pures Appl. **4** (1959), 725–729.

Dans l'article précédent [voir l'analyse ci-dessus] l'auteur a montré qu'un théorème de S. Stoilow [cf. S. Marcus, même Rev. **2** (1957), 409–412; MR **20** #4616] concernant les fonctions définies sur un intervalle (a, b) et continues, reste valable si on substitue à la continuité la propriété de Darboux. Dans le présent article sont données quelques conséquences de ce résultat. En voici un spécimen. f désigne une fonction sur (a, b), M sa borne supérieure, m sa borne inférieure, D l'ensemble des points x de (a, b) où f admet une dérivée finie ou infinie, $f(D)$ l'image de D par f. Alors: si f jouit de la propriété de Darboux, une condition nécessaire et suffisante pour que $f(D)$ diffère de $[m, M]$ par un ensemble de mesure nulle est que, pour presque tout t de $[m, M]$, l'ensemble $E_t = \{x : f(x) = t\}$ admette un point isolé.

Chr. Y. Pauc (Nantes)

6875:

Ionescu, D. V. Le théorème fondamental sur les fonctions implicites. Inst. Politehn. Cluj. Lucrări Şti. **1** (1958), 57–67. (Romanian. Russian and French summaries)

Author's summary: "On y présente une nouvelle démonstration, dans laquelle la méthode habituelle de l'induction complète est remplacée par une méthode analogue à l'algorithme de Gauss, de la résolution des systèmes d'équations linéaires."

6876:

Allanson, J. T. A note on rational polynomials. Les mathématiques de l'ingénieur, pp. 185–187. Mém. Publ. Soc. Sci. Arts Lett. Hainaut, vol. hors série, 1958.

The author proves that if a and b are real, then any given polynomial is the difference of two other polynomials the roots of which all lie in the interval (a, b).

D. H. Lehmer (Berkeley, Calif.)

6877:

Kuczma, M. Note on convex functions. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 25–26.

Let E be a subset of a (real or complex) linear space X, and f a function from X into the reals such that $f(\frac{1}{2}(x+y))$

$\leq \frac{1}{2}(f(x)+f(y))$ for every x, y. Let $\hat{E}$ be the smallest subset S of X such that (*) $S \supseteq E$ and (**) $x \in S$, $y \in S$ imply $\frac{1}{2}(x+y) \in S$ (i.e., $\hat{E}$ is the intersection of all subsets S of X satisfying (*) and (**)). Then every upper bound of f on E is also an upper bound of f on $\hat{E}$.

O. Shisha (Washington, D.C.)

6878:

Wahde, Gösta. An extremal problem related to the theory of quasi-analytic functions. Math. Scand. **7** (1959), 126–132.

The author solves the following two minimal problems for real functions: Given an even integral function $H(x) = \sum_{\nu=0}^{\infty} x^{2\nu}/m_\nu^2$ (m_ν real, $m_0 = 1$), let $\alpha_1 = \inf \int_0^\infty H(x)f(x)^2 dx$ for all real f such that $\int_0^\infty f(x)dx = (\frac{1}{2}\pi)^{1/2}$, $\int_0^\infty x^{2\nu} f(x)dx = 0$ ($\nu \geqq 1$); and

$$\alpha_2 = \inf \sum_{\nu=0}^{\infty} \frac{1}{m_\nu^2} \int_0^\infty f^{(\nu)}(x)^2 \, dx$$

for all real f such that $f(0) = 1$, $f^{(\nu)}(0) = 0$ ($\nu \geqq 1$). Then

$$\alpha_1 = \alpha_2 = \frac{1}{\pi} \int_0^\infty \frac{1}{x^2} \log H(x) \, dx.$$

This result furnishes a new proof of the main theorem in the theory of quasi-analytic functions.

K. Strebel (Fribourg)

6879:

Kopeć, J.; Musielak, J. On quasianalytic classes of functions, expansible in series. Ann. Polon. Math. **7** (1960), 285–292.

Les auteurs disent que $\alpha(x)$ ($x > 0$) satisfait la condition (M) si $\alpha(x)$ est dérivable, et si $q(x) = x\alpha'(x)$ croit vers l'infini avec x. Les auteurs démontrent plusieurs théorèmes dont voici un exemple [ces théorèmes généralisent un théorème de Mandelbrojt, *Séries de Fourier et classes quasi-analytiques de fonctions*, Gauthier-Villars, Paris, 1935]: Soit $\varphi \in C\{1\}$ dans $(-\infty, \infty)$, et soit $\{\lambda_n\}$ une suite de nombres avec $\inf \lambda_n > 0$. Supposons que $\alpha(x)$ satisfait la condition (M) et que $\int^\infty [\alpha(x)/x^2]dx = \infty$. Soit remplie l'une des conditions (1) $\{\lambda_n^{-1}\} \in l^p$, pour un certain $p > 0$, et $|a_n| < \exp[-\alpha(\lambda_n)]$, (2) il existe deux fonctions positives $\nu(x)$ et $\omega(x)$ telles que $|a_n| < 1/\nu(\lambda_n)$, $\sum 1/\omega(\lambda_n) < \infty$, $\alpha(x) = \log(\nu(x)/\omega(x))$, alors la fonction $\sum a_n \varphi(\lambda_n x) = f(x)$ appartient à la classe quasi-analytique $C\{m_{nl}\}$ définie de la façon suivante (pour chaque $l > 0$ fixe): $m_{nl} = \Phi_{n+l}(x_{n+l})$, où x_k est défini par $q(x_k) = k$, $\Phi_k(x) = x^k \exp[-\alpha(x)]$.

S. Mandelbrojt (Paris)

MEASURE AND INTEGRATION

See also 6742, 6743, 6744, 6955, 6990.

6880:

Leĭfman, L. Ya. On convergence of integrals depending on a parameter in an abstract space. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 824–827. (Ukrainian. Russian and English summaries)

Soient R un ensemble, $\mathscr{B}$ un corps borélien de parties de R et μ une mesure sur $\mathscr{B}$; T un espace topologique vérifiant l'axiome T_1 de séparabilité, $A \subset T$ et $\alpha_0 \in T$ un point d'accumulation de A. Soit $E \in \mathscr{B}$ et pour chaque $\alpha \in A$, soit φ_α une fonction définie sur $\mathscr{B}E$, absolument continue (par rapport à μ). L'A. démontre le résultat suivant: si pour tout $G \in \mathscr{B}E$ il existe $\lim_{\alpha \to \alpha_0} \varphi_\alpha(G) = \Phi(G)$ et si Φ est absolument continue, alors il existe un voisinage U de α_0 tel que la famille $(\varphi_\alpha)_{\alpha \in A \cap U}$ soit uniformément absolument continue.

N. Dinculeanu (Bucharest)

6881:

Matthes, Klaus. Über eine Verallgemeinerung des Lebesgueschen Integralbegriffes. IV. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **8** (1958/59), 331–338. (Russian, English and French Summaries)

In den drei ersten Teilen seiner Arbeit [dieselbe Z. **5** (1955/56), 287–295; **6** (1956/57), 221–236; **7** (1957/58), 329–344; MR **19**, 1042; **20** #5270; **21** #5713] hatte der Verf. mehrfach von ordnungstheoretischen Sätzen aus Birkhoff, *Lattice theory*, rev. ed., [Amer. Math. Soc., New York, 1948; MR **10**, 673], Gebrauch gemacht, die sich letzten Endes auf die Gleichung $\tau_{M \times M} = \tau_M \times \tau_M$ über die Ordnungstopologie τ_M einer geordneten Menge M stützen, welche aber nicht uneingeschränkt gilt. Er stellt nun fest, daß er in Wirklichkeit nur eine abgeschwächte Distributivitätseigenschaft benötigt und nimmt dies zum Anlaß, solche Eigenschaften in allgemeinerer Form eingehend zu untersuchen. Zu jeder unendlichen Kardinalzahl α werden der Begriff eines α-regulären Verbandes durch eben eine Distributivitätseigenschaft definiert und damit Erweiterungssätze über Homomorphismen Boolescher Algebren aus Teil II neu formuliert und bewiesen.—Eine geordnete Menge M, die $\tau_{M \times M} = \tau_M \times \tau_M$ erfüllt, heißt topologisch. Dies ist im Fall einer σ-vollständigen Booleschen Algebra oder auch eines bedingt σ-vollständigen Vektorverbandes hinreichend für die $\aleph_0$-Regularität. Andererseits ist z.B. jede Maßalgebra topologisch, so daß darauf die betreffenden Erweiterungssätze anwendbar sind. Schließliche wird die Erweiterung einer abzählbar additiven Abbildung einer Halbalgebra in die Menge der positiven Vektoren eines $\aleph_0$-regulären bedingt σ-vollständigen Vektorverbandes V zu einem V-Maß behandelt.

K. Krickeberg (Heidelberg)

6882:

Tsujimoto, Hitoshi. On the sets of regular measures. I, II. Proc. Japan Acad. **35** (1959), 273–278, 372–377.

§ 1. $(X, \mathbf{S})$: topological measurable space. E: generic subset of $\mathbf{S}$. μ, μ_i ($i = 1, 2, \cdots$), μ_λ ($\lambda \in \Lambda$), ν: measures on $\mathbf{S}$, taking possibly infinite values. $\mathbf{C}$: class of the compact measurable sets. $\mathbf{U}$: class of the open measurable sets. E is called "inner μ-regular" if $\mu(E) = \sup\{\mu(C): E \supseteq C, C \in \mathbf{C}\}$, "outer μ-regular" if $\mu(E) = \inf\{\mu(U): E \subseteq U, U \in \mathbf{U}\}$. If every measurable set is inner [outer] regular, μ is called inner [outer] regular. §§ 2, 3, 4. Assuming a regularity property for the measures μ_i, μ_λ, $\nu_j = \int f_j d\mu$ ($j = 1, 2$), the author investigates the validity of the same property for the measure ν defined by (§ 2) $\nu(E) = \lim \mu_i(E)$ provided the limit exists for each E, (§ 3) $\nu = \sup \mu_\lambda$ or $\nu = \inf \mu_\lambda$, (§ 4) $\nu(E) = \int_E (f_1 f_2)^{1/2} d\mu$. § 5 deals with the supremum, infimum and direct sum of two irregular measures. Among the numerous propositions and illustrative counterexamples we quote these as typical: Theorem 3(1). If a set E is inner [outer] regular with respect to μ_1 and μ_2, then it is with respect to $\nu = \sup\{\mu_1, \mu_2\}$. Theorem 4(1). If every μ_λ is outer regular, then $\inf \mu_\lambda$ is also outer regular; Example 2. X_1 is a countable set $\{a_i\}$, X_2 a noncountable set, disjoint from X_1. Let $X = X_1 \cup X_2$ be a

concrete space, S the Boolean σ-algebra generated by $\{a_1\}, \cdots, \{a_i\}, \cdots$, and X_2, μ_1 and μ_2 defined by $\mu_1(\{a_i\}) = i^{-1}$ and $\mu_2(\{a_i\}) = i^{-2}$ for odd i, $\mu_1(\{a_i\}) = i^{-2}$ and $\mu_2(\{a_i\}) = i^{-1}$ for even i, $\mu_1(X_2) = \mu_2(X_2) = c$, $0 < c \leqq \infty$. μ_1 and μ_2 are outer regular at X_1 but $\nu = \inf\{\mu_1, \mu_2\}$ is not. The proofs are standard. Pathological situations arise in connection with the presence of some non-σ-finite set E. {Remarks by the reviewer. (i) The reviewer does not feel sure about the meaning of "concrete space" X in Example 2. If it means the topological space admitting as open sets X and the empty subset exclusively, then the non-countability assumption on X_2 would seem to be superfluous. (ii) Apparently the only property of $\mathbf{U}[\mathbf{C}]$ used is the finite intersection [union] stability.}

Chr. Y. Pauc (Nantes)

6883:

Očan, Yu. S. Representation of functions of bounded variation by means of a generalized integral. Moskov. Gos. Ped. Inst. Uč. Zap. **138** (1958), 11–45. (Russian)

Soit f une fonction positive croissante sur $(c, +\infty)$; soit φ une fonction sommable (au sens de Lebesgue) sur $[a, b]$. Pour $\alpha > 0$, $\beta > 0$, $t > c$, considérons la fonction $\varphi_{-\alpha t, \beta t}$ définie sur $[a, b]$, égale à $\varphi(x)$ si $-\alpha t \leqq \varphi(x) \leqq \beta t$, à $-\alpha t$ si $\varphi(x) < -\alpha t$ et à βt si $\varphi(x) > \beta t$. On dit que φ est intégrable (au sens généralisé) sur $[a, b]$ si $(1/f(t)) \times \int_a^b \varphi_{-\alpha t, \beta t}\, dx$ a une limite finie l quand $t \to +\infty$, indépendente de α et β. La limite l est appelée l'intégrale généralisée de φ sur $[a, b]$ et est notée $\int_a^{*b} \varphi(x)\, dx$. L'ensemble des fonctions intégrables est un espace vectoriel. On dit que deux fonctions sont équivalentes si l'intégrale de leur différence est nulle sur chaque intervalle $[x_1, x_2] \subset [a, b]$. Désignons par Φ l'ensemble des fonctions de la forme $\varphi = \varphi_1 - \varphi_2$, où φ_1 et φ_2 sont des fonctions positives intégrables sur chaque intervalle $[x_1, x_2] \subset [a, b]$, et par U l'ensemble des fonctions u à variation bornée sur $[a, b]$. En identifiant deux fonctions équivalentes de Φ et deux fonctions de U dont la différence est constante, l'A. montre que, si f vérifie certaines conditions supplémentaires, il y a une correspondence biunivoque entre Φ et U donnée par l'égalité $\int_a^{*x} \varphi(x)\, dx = u(x) - u(a)$. Alors, pour toute fonction réelle continue F sur $[a, b]$ on a $\int_a^{*b} F(x)\varphi(x)\, dx = \int_a^b F(x)\, du(x)$, où l'intégrale du second membre est au sens de Stieltjes.

N. Dinculeanu (Bucharest)

6884:

Marcus, Solomon. Sur une théorie du type Lebesgue pour l'intégrale de Riemann. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2** (**50**) (1958), 175–184.

Cette note est un résumé de l'article de l'auteur ayant le même titre et paru aux Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 333–369 [MR **21** #3529]. La note de O. Haupt et Chr. Pauc [Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1955**, 347–369; MR **18**, 198] pourrait figurer dans la bibliographie.

Chr. Y. Pauc (Nantes)

6885:

Goffman, Casper. Non-parametric surfaces given by linearly continuous functions. Acta Math. **103** (1960), 269–291.

A function $f(x, y)$ defined on the unit square I is linearly continuous if f is continuous in x for almost every y and in y for almost every x. If f_n tends to f uniformly on almost every line parallel to the x- or the y-axis, then f_n is said to converge linearly to f on I. By lower semicontinuity an area is defined on the space L of linearly continuous functions, which is shown to agree on L with area as defined on the wider class of measurable functions by Cesari [Ann. Scuola Norm. Sup. Pisa (2) **5** (1936), 299–313] and by the author [Rend. Circ. Mat. Palermo (2) **2** (1954), 203–235; MR **16**, 457]. The main theorem is that the area of a function in L equals the Hausdorff 2-dimensional measure of its graph, provided one excludes points on the graph corresponding to a certain subset of I which is the cartesian product of null sets on the x- and the y-axes. This extends a theorem of Federer for continuous f [Trans. Amer. Math. Soc. **62** (1947), 114–192; MR **9**, 231]. To any f in L there corresponds a measure μ_f on I, agreeing on open rectangles R with the area of f restricted to R, such that if f and g have finite area and $f = g$ on a Borel set E, then $\mu_f(E) = \mu_g(E)$. For continuous f this result was proved by Verčenko [Mat. Sb. (N.S.) **10** (**52**) (1942), 11–32; MR **4**, 154].

W. H. Fleming (Providence, R.I.)

FUNCTIONS OF COMPLEX VARIABLES

See also 6867, 6998, 7244.

6886:

Položiĭ, G. N. (p, q)-analytic functions of a complex variable and some of their applications. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 483–515. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

Let $f(z) = u + iv$ be single-valued in the simply-connected domain G of the z-plane. Then $f(z)$ is said to be (p, q)-analytic in G if u, v are continuously differentiable solutions of the system of equations

$$(1) \qquad pu_x + qu_y - v_y = 0, \quad -qu_x + pu_y + v_x = 0,$$

where $p = p(x, y)$, $q = q(x, y)$ are single-valued and continuous in G and p is positive there. The case $q \equiv 0$ corresponds to the p-analytic functions, previously studied by the author in considerable detail in earlier papers [see, e.g., Dokl. Akad. Nauk SSSR **58** (1947), 1275–1278; **60** (1948), 769–772; Mat. Sb. (N.S.) **24** (**66**) (1949), 375–384; **32** (**74**) (1953), 485–492; MR **9**, 507; **10**, 698; **11**, 171; **15**, 320]. These latter functions are a generalization of the sigma-monogenic functions originally investigated by Bers and Gelbart [Quart. Appl. Math. **1** (1943), 168–188; Trans. Amer. Math. Soc. **56** (1944), 67–93; MR **5**, 25; **6**, 86].

In the present paper, partly in the nature of a survey of his own work and related work by others, the author continues work on (p, q)-analytic functions initiated in an earlier paper [Trudy Tret'ego Vsesoyuznogo Mat. S"ezda, Tom 1, pp. 95–96, Akad. Nauk SSSR, Moscow, 1956; see MR **20** #6973a]. The line integral $\int_{z_0}^{z} f(z)dz$, taken along a curve lying in G, is defined by $\int_{z_0}^{z} u d\tilde{Z} + iv dZ$, where Z, $\tilde{Z}$ are so-called "conjugate variables", i.e., $Z = X + iY$, $\tilde{Z} = \tilde{X} + i\tilde{Y}$ are such that $Z^* = X + i\tilde{Y}$ and $\tilde{Z}^* = -Y + i\tilde{X}$ are twice continuously differentiable (p, q)-analytic functions in G. Analogues of Cauchy's theorem and formula,

for ordinary analytic functions, are then stated for (p, q)-analytic functions. The line integral $\int_{z_0}^{z} f(z)dz$ of a (p, q)-analytic function will again be (p, q)-analytic if and only if p and q are functions of an arbitrary harmonic function $\beta(x, y)$, in which case we have the representation

$$(2)\quad \int_{z_0}^{z} f(z)dz = \int_{z_0}^{z} u\left(d\alpha+\frac{q}{p}\,d\beta\right)-v\,\frac{d\beta}{p} + i\int_{z_0}^{z} v\left(d\alpha-\frac{q}{p}\,d\beta\right)+\frac{u}{p}\,(p^2+q^2)d\beta,$$

where α and β are harmonic conjugates.

The "(p, q)-derivative with respect to z" of $f(z)=u+iv$ (not necessarily (p, q)-analytic) is defined by

$$(3)\quad \frac{'df}{dz} = \frac{1}{2}\left(\frac{\partial u}{\partial\alpha}-\frac{q}{p}\frac{\partial u}{\partial\beta}+\frac{1}{p}\frac{\partial v}{\partial\beta}\right)+i\frac{1}{2}\left(\frac{\partial v}{\partial\alpha}+\frac{q}{p}\frac{\partial v}{\partial\beta}-\frac{p^2+q^2}{p}\frac{\partial u}{\partial\beta}\right)$$

and the "(p, q)-derivative with respect to $\bar{z}$" by

$$\frac{'df}{d\bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial\alpha}+\frac{q}{p}\frac{\partial u}{\partial\beta}-\frac{1}{p}\frac{\partial v}{\partial\beta}\right)+i\frac{1}{2}\left(\frac{\partial v}{\partial\alpha}-\frac{q}{p}\frac{\partial v}{\partial\beta}+\frac{p^2+q^2}{p}\frac{\partial u}{\partial\beta}\right),$$

where $p=p(\beta)$, $q=q(\beta)$, and $\alpha+i\beta$ is an analytic function of z. From these definitions it follows that $f(z)$ will be (p, q)-analytic if and only if $'df/d\bar{z}=0$. Moreover, if $f(z)$ is (p, q)-analytic so is $'df/dz$, and (3) becomes $'df/dz=\partial u/\partial\alpha+i\partial v/\partial\alpha$. Employing these definitions the author comments briefly on "elementary (p, q)-analytic functions", defined as the solutions of particular (p, q)-analytic differential equations. He concludes by applying the concepts of differentiation and integration of p-analytic functions to solving specific second and fourth order partial differential equations and illustrates the methods involved with a detailed discussion of two problems from the theory of elasticity. There is an extensive bibliography. *J. F. Heyda* (Cincinnati, Ohio)

6887:

Lange, L. J.; Thron, W. J. A two-parameter family of best twin convergence regions for continued fractions. Math. Z. **73** (1960), 295–311.

The principal result is the following. If $|a|<\rho<|1+a|$ and

$$a_{2n-1} = c_{2n-1}^2,\quad c_{2n-1}\in C_1(ia, \rho): |z\pm ia| \le \rho,$$
$$a_{2n} = c_{2n}^2,\quad c_{2n}\in C_2(i(1+a), \rho): |z\pm i(1+a)| \ge \rho,$$

then the continued fraction

$$1+\frac{a_1}{1}+\frac{a_2}{1}+\cdots$$

converges. If a is real the value of the continued fraction lies in the circular disc $|z-1-a|\le\rho$, and a statement about uniform convergence can be made. C_1, C_2 are best twin convergence regions in the sense that for twin convergence regions C_1^*, C_2^* with $C_1\subset C_1^*$, $C_2\subset C_2^*$, it follows that $C_1^*=C_1$, $C_2^*=C_2$. The non-uniqueness of best twin convergence regions for given C_1 is noted by oberserving that C_1, C_2', where $C_2'=C_2(i(1-a), \rho)$, $|\mathrm{Re}\, a|<\frac{1}{2}$, are also best twin convergence regions. The case $a=0$ yields a result of Singh and Thron [Proc. Amer. Math. Soc. **7** (1956), 277–282; MR **17**, 1076].

For real a, the proof of the theorem depends on the method of nested circles, and the statements on the value and on the uniform convergence of the continued fraction are natural consequences of the method. The extension to complex a uses the result for real a and the Stieltjes-Vitali theorem. The fact that the regions are best is established by considering suitable 2-periodic and 4-periodic continued fractions. *W. T. Scott* (Evanston, Ill.)

6888:

Ferlan, Nives Maria; Skof, Fulvia. Ancora sulla permanenza della struttura lacunare nel prolungamento analitico. Ann. Mat. Pura Appl. (4) **49** (1960), 193–212. (English summary)

Continuation de l'étude commencée dans Riv. Mat. Univ. Parma **8** (1957), 345–360 [MR **21** #7297].
S. Mandelbrojt (Houston, Tex.)

6889:

Sadowska, D. Sur un problème aux limites de la théorie des fonctions analytiques. Ann. Polon. Math. **8** (1960), 193–200.

Let $L_0, L_1, \cdots, L_p$ be Jordan closed curves in the complex plane without points in common, L_0 containing $L_1, \cdots, L_p$. L_i contains the domain S_i^-, $1\le i\le p$, S_i^- disjoint; S_0^- is the infinite domain bounded by L_0; and S^+ is the domain bounded by $L_0, \cdots, L_p$. A solution to the following problem is obtained. To find a function $\Phi(z)$ analytic separately in each of the domains $S^+, S_0^-, \cdots, S_p^-$ and for which the limit values at each point $t\in L=L_0+L_1+\cdots+L_p$ satisfy

$$\Phi^+(t) = G(t)\Phi^-(t)+g(t)+\int_L \frac{N_1(t, \tau)}{|t-\tau|^{\alpha_1}}\,\Phi^+(\tau)\,d\tau + \int_L \frac{N_2(t, \tau)}{|t-\tau|^{\alpha_2}}\,\Phi^-(\tau)\,d\tau,$$

where $\Phi^+(t)$, $t\in L$, is the limit value of Φ with respect to S^+ and $\Phi^-(t)$ is the limit value with respect to $S^-=S_0^-+\cdots+S_p^-$. $G(t)$, $g(t)$ are complex functions, $G(t)\ne0$ on L, and $0<\alpha_i<1$, $i=1, 2$. The problem is solved with the hypothesis that $g(t)$, $G(t)$, $N_1(t, \tau)$, $N_2(t, \tau)$ satisfy suitable Hölder inequalities on L and with the assumption that the curves $L_0, \cdots, L_p$ have a continuous tangent at each of their points.
M. S. Robertson (New Brunswick, N.J.)

6890:

Mandžavidze, G. F. Approximate solution of boundary problems of the theory of analytic functions. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 365–370. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

Let L be a simple, smooth, closed contour in the complex plane with interior D^+ and exterior D^-, on which are defined matrices $G(t)=(G_{ik}(t))$ and $F(t)=(F_{ik}(t))$ whose elements are Hölder continuous on L with index μ; in addition it is assumed that $\det G(t)\ne0$ on L. Consider the problem of finding a sectionally-holomorphic matrix $\Phi(z)$ with elements of finite degree at infinity whose boundary values $\Phi^+(t)$, $\Phi^-(t)$, along the inner and outer edges of L, respectively, satisfy (1) $\Phi^+(t)=G(t)\Phi^-(t)+F(t)$. The solution of this problem in terms of Cauchy integrals is given by Mushelišvili in his well-known book *Singular integral equations* [OGIZ, Moscow-Leningrad, 1946; English translation, Noordhoff, Groningen, 1953; MR **8**, 586; **15**, 434]. The author determines this solution approximately by

the following scheme. Define the sectionally-holomorphic matrix $\varphi(z)$ to be $\Phi(z)$ for z in D^+ and $R(z)\Phi(z)$ for z in D^-, where $R=(R_{ik})$ is a matrix whose elements are rational functions such that on L one has $\|G_{ik}-R_{ik}\|_\nu<\varepsilon$ ($i, k=1, 2, \cdots, n$), where ε is a small positive number and the norm is defined by

$$(2)\qquad \|f\|_\nu = \max_{t\in L}|f(t)|+\sup\frac{|f(t_1)-f(t_2)|}{|t_1-t_2|^\nu}\qquad (0<\nu<1,\ \nu<\mu).$$

The boundary relation (1) can then be rewritten as

$$(3)\qquad \varphi^+(t)-\varphi^-(t) = g(t)\varphi^-(t)+F(t)$$

where $g=(G-R)R^{-1}$. (ε is taken sufficiently small so that $\det R(t)\neq 0$ on L.) It is then shown, for the sequence of sectionally-holomorphic matrices $\varphi_m(z)$ defined recursively by

$$(4)\qquad \varphi_m(z) = \frac{1}{2\pi i}\int_L\frac{g(t)\varphi_{m-1}^-(t)dt}{t-z}+\frac{1}{2\pi i}\int_L\frac{F(t)dt}{t-z},$$

that the sequences $\varphi_m^+(t)$, $\varphi_m^-(t)$ converge in the sense of the norm (2) whenever ε is small enough, and hence $\varphi_m(z)\to\varphi(z)$. As a corollary of the method the author constructs an approximate solution of the homogeneous problem $\alpha^+(t)=G(t)\alpha^-(t)$ with poles possible only in D^-, whence by known methods the canonical solution may be obtained. The remainder of the paper is devoted to a modification of the method to allow $G(t)$ and $F(t)$ to have at most a finite number of jump points on L.

J. F. Heyda (Cincinnati, Ohio)

6891:

Koseki, Ken'iti. Über die Koeffizienten der schlichten Funktionen. Math. J. Okayama Univ. 9 (1959/60), 173–197.

A proof of Bieberbach's conjecture is claimed. {The conditions on the function $\alpha(t)$ at the top of page 190 are, however, impossible of fulfillment, which invalidates the subsequent argument.} *Herbert S. Wilf* (Urbana, Ill.)

6892:

Tumarkin, G. C.; Havinson, S. Ya. Mutual orthogonality of boundary values of certain classes of analytic functions in multiply connected domains. Uspehi Mat. Nauk 14 (1959), no. 3 (87), 173–180. (Russian)

Let G be a finitely-connected domain of the z-plane bounded by simple closed rectifiable curves γ_j ($j=1, \cdots, n$). We say that two classes of complex-valued functions K_1 and K_2 defined on $\Gamma=\sum\gamma_j$ are "mutually orthogonal" if $\int_\Gamma f_1(\zeta)f_2(\zeta)d\zeta=0$ holds for every pair of functions $f_1(\zeta)\in K_1$ and $f_2(\zeta)\in K_2$, and if the condition that some function $f(\zeta)$ is orthogonal to one of the two classes always entails that $f(\zeta)$ belongs to the other class. We say that the analytic functions $f(z)$ are of class $B(G)$ if they are bounded in G and of class $E_p(G)$ ($p>0$) if

$$\limsup_{j\to\infty}\int_{\Gamma^j}|f(z)|^p|dz|<\infty$$

along some sequence of closed rectifiable curves $\{\Gamma^j\}$ converging to Γ inside G. Denote by $B(\zeta)$ and $E_p(\zeta)$ the classes of function defined on Γ by boundary values of functions of $B(G)$ and $E_p(G)$, respectively.

The work is concerned with mutual orthogonalities between these classes of functions. It contains, for instance, the results that (i) the classes $B(\zeta)$ and $E_1(\zeta)$ are mutually orthogonal, and (ii) the classes $E_p(\zeta)$ and $E_q(\zeta)$ ($1/p+1/q=1$) are likewise. It shows one of the obtained results may be applied to solve a problem on approximation of qth power summable functions on Γ by a certain system of rational functions. *T. Kubo* (Kyoto)

6893:

Szegö, G. Recent advances and open questions on the asymptotic expansions of orthogonal polynomials. J. Soc. Indust. Appl. Math. 7 (1959), 311–315.

After a short account of recent and more classical results on this field, mainly due to the author, some unsolved problems of outstanding interest are stated: Let C be a closed curve in the complex plane consisting of a finite number of analytic arcs, each (outer) angle of intersection lying between 0 and 2π. Let $\{p_n(x)\}$ be the sequence of polynomials orthogonal on C with the weight $w(x)=|D(z)|^2$, where $D(z)$ is a regular, nonvanishing function inside the unit circle, and $x=\psi(z)$ maps $|z|<1$ onto the exterior of C and $\psi(0)=\infty$. The author's conjectures are as follows: The asymptotic behaviour of $p_n(x)$ for $n\to\infty$ is determined essentially by the local data near x; it is not influenced by multiplying $w(x)$ by a factor bounded away from zero and infinity; it is, in a certain well-defined sense, a conformal invariant.

G. Freud (Budapest)

6894:

Browder, Felix E. On the proof of Mergelyan's approximation theorem. Amer. Math. Monthly 67 (1960), 442–444.

Mergelyan a montré [Dokl. Akad. Nauk SSSR 78 (1951), 405–408; MR 13, 23] que si un compact K ne divise pas le plan, toute fonction continue sur K et analytique à l'intérieur de K est limite uniforme de polynomes. L'auteur simplifie la démonstration de Mergelyan en établissant de manière élémentaire des inégalités sur les coefficients d'une fonction $F(w)=aw+b+\sum_1^\infty a_jw^{-j}$ appliquant $|w|>1$ dans le complémentaire d'un compact de diamètre convenable. *J.-P. Kahane* (Montpellier)

6895:

Al'per, S. Ya. Approximation of analytic functions in the mean over a region. Dokl. Akad. Nauk SSSR 136 (1961), 265–268 (Russian); translated as Soviet Math. Dokl. 2, 36–39.

The author considers the class H_p' ($p\geq 1$) of functions analytic in a bounded simply-connected region D with

$$(*)\qquad \|f\| = \left\{\iint_D|f(z)|^p\,dx\,dy\right\}^{1/p}<\infty.$$

He introduces

$$\omega_p(\delta, f) = \sup_{|h|\leq\delta}\left\{\iint_D|f(z+\lambda)-f(z)|^p\,dx\,dy\right\}^{1/p},$$

and $\rho_n^{(p)}(f, D)$, the best approximation to f by polynomials of degree n in the norm (*). He gives the following theorems, in which A denotes various constants. (1) A

necessary and sufficient condition for $f \in H_p'$ in $|z|<1$ to satisfy

$$\left\{\iint |f(re^{i(\theta+\tau)})-f(re^{i\theta})|^p r\,dr\,d\theta\right\}^{1/p} \leqq A|\tau|^\alpha,$$

$0<\alpha\leqq 1$, is that

$$\left\{\int_0^{2\pi}\int_0^\rho |f'(re^{i\theta})|^p r\,dr\,d\theta\right\}^{1/p} \leqq A(1-\rho)^{\alpha-1}, \quad 0<\rho<1.$$

(2) If D is bounded by an analytic Jordan curve, if $f^{(m)}(z)\in H_p'$, and if $\omega_p(\delta, f^{(m)})\leqq A\delta^\alpha$, $0<\alpha\leqq 1$, then $\rho_n^{(p)}(f, D)\leqq An^{-m-\alpha}$ for every integer $n\geqq 1$. (3) Conversely, if the last inequality holds for every n then $f^{(m)}(z)\in H_p'$ and $\omega(\delta, f^{(m)})\leqq A\delta^\alpha$ if $0<\alpha<1$, $\omega_p(\delta, f^{(m)})\leqq A\delta(1+|\log\delta|)$ for $\alpha=1$. Next the author gives necessary and sufficient conditions for $f^{(m)}$ to belong to an integrated Lipschitz class on the boundary of D. Finally, returning to a general D with simply-connected complement, he gives an upper bound for $\rho_n^{(p)}(f, D)$ in terms of $\omega_p(\lambda(n), f)$, where $\lambda(n)$ depends on D.

R. P. Boas, Jr. (Evanston, Ill.)

6896:

Al'per, S. Ya. Approximation in the mean of analytic functions of class E_p. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 273–286. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The author considers mean pth power approximation of boundary values of analytic functions defined in a bounded domain D with a sufficiently smooth boundary Γ. The smoothness condition imposed on Γ is the following: Let Γ have a tangent everywhere. The tangent to Γ at 's' (s = arclength) makes an angle $\theta(s)$ with the positive real axis. The modulus of continuity $j(u)$ of $\theta(s)$ satisfies $\int_0 (j(u)/u)du<\infty$. Typical results are the following. Theorem 1: If $f(z)\in E_p(D)$ $(p>1)$, i.e.,

$$\int_0^{2c} |f(\psi(re^{i\theta}))|^p|\psi(re^{i\theta})|d\theta<K$$

($r<1$; $z=\psi(w)$ function mapping $|w|<1$ on D), then there is a polynomial $p(z)$ of degree n so that

$$\int_\Gamma |f(z)-p(z)|^p|dz| < A(p, D)w(1/n, f_0),$$

where $w(\delta, f_0)$ is the generalized modulus of continuity of the function $f_0(e^{i\theta})$ defined by $f_0(e^{i\theta})=f[\psi(e^{i\theta})]$,

$$w(\delta, f_0) = \sup_{|h|\leqq\delta}\left(\int_0^{2c} |f_0(e^{i\theta+ih})-f_0(e^{i\theta})|^p d\theta\right)^{1/p}.$$

Theorem 2': If $f(z)\in L_p(\Gamma)$ and if to every positive integer n there is a polynomial of degree n such that

$$\left(\int_\Gamma |f(z)-p(z)|^p|dz|\right)^{1/p} \leqq Kn^{-r-\alpha}$$

(r non-negative integer, $0<\alpha\leqq 1$), then $f(z)$ is the boundary function of a function $f\in E_p(D)$, and if on Γ $f^{(r)}(z)=f_r(\psi(e^{i\theta}))$, then

$$w_p(\delta, f_r) \leqq K\delta^\alpha \quad (0<\alpha<1),$$

$$w_p(\delta, f_r) \leqq K\delta(1+|\log\delta|) \quad (\alpha=1).$$

W. H. J. Fuchs (Ithaca, N.Y.)

6897:

Gröbner, Wolfgang. ★Die Lie-Reihen und ihre Anwendungen. Mathematische Monographien, 3. VEB Deutscher Verlag der Wissenschaften, Berlin, 1960. vii+112 pp. DM 22.00.

Un symbole de transformation infinitésimale

$$D \equiv \theta_1\frac{\partial}{\partial z_1}+\theta_2\frac{\partial}{\partial z_2}+\cdots+\theta_n\frac{\partial}{\partial z_n}$$

(ou, pour abriger, $\theta_i\partial/\partial z_i$), où les θ sont des fonctions analytiques données des z, conduit, comme ou sait, à faire correspondre à toute analytique $f(z_1, z_2, \cdots, z_n)$ la série

$$f(z)+tDf(z)+\cdots+\frac{t^n}{n!}D^nf(z)+\cdots$$

qu'il est naturel de représenter par le symbole $e^{tD}f$. C'est à de tables séries que l'auteur propose de donner le nom de séries de Lie. Il étudie leurs propriétés: convergence, dérivation par rapport à t, par rapport à un des z, $\cdots$, règle de permutation: en posant $Z_i=e^{tD}z_i$, $\Psi=\psi_i(z)\partial/\partial z_i$, $\Psi_*=\psi_i(Z)\partial/\partial Z_i$, on a $\Psi_*F(Z)=e^{tD}(\Psi F(z))$.

Il montre leur utilité dans la théorie des systèmes différentiels, et, en particulier, des systèmes différentiels de la mécanique céleste. Il les applique également à l'inversion des systèmes de fonctions.

C'est d'ailleurs la géométrie algébrique qui dès 1938 avait conduit l'auteur à essayer d'applique systématiquement ces séries de Lie. L'ouvrage qu'il donne aujourd'hui se termine par un chapître où il montre leur utilité pour le paramétrisation des multiplicités algébriques.

M. Janet (Paris)

6898:

Eremin, S. A. Complete systems and bases in spaces of analytic functions of several complex variables. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 305–315. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The author proves multi-dimensional analogues of one-variable theorems on complete systems of translations, derivatives, etc. He considered problems of the same kind previously [Ukrain. Mat. Ž. **8** (1956), 361–376; MR **18**, 798], but only in polycylinders, whereas here he considers more general regions. *R. P. Boas, Jr.* (Evanston, Ill.)

6899:

Fuks, B. A. Kernel functions of regions. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 287–294. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

This paper gives a brief account of the theory of S. Bergmann's kernel function, including some recent results, but no new results. *H. Tornehave* (Copenhagen)

SPECIAL FUNCTIONS

6900:

Frank, Evelyn. A new class of continued fraction expansions for the ratios of Heine functions. II. Trans. Amer. Math. Soc. **95** (1960), 17–26.

[Pour la première partie, voir mêmes Trans. **88** (1958),

288–300; MR **20** #4017.] Poursuivant ses recherches sur les développements en fractions continues, l'auteur applique sa méthode aux fonctions $\Phi(a, b, c, q, z)/\Phi(a+1, b+1, c+1, q, z)$ et à des fonctions analogues, où Φ désigne la fonction de Heine. Elle en tire d'intéressantes identités pour des valeurs particulière des paramètres.
R. Campbell (Caen)

6901:
Frank, Evelyn. A new class of continued fraction expansions for the ratios of Heine functions. III. Trans. Amer. Math. Soc. **96** (1960), 312–321.

En continuation de #6900, l'auteur établit l'équivalence de certains de ces développements, et donne des formules relatives aux réduites de ces fonctions continues.
R. Campbell (Caen)

6902:
Tschauner, Johann. Über einen Zusammenhang zwischen den Kugelfunktionen und Zylinderfunktionen. Math. Z. **73** (1960), 457–459.

L'auteur donne une démonstration intéressante du développement générateur

$$(-1)^m t^{-m} \exp(t \cos\theta) J_m(t \sin\theta) = \sum_{n=0}^{\infty} \frac{P^m_{n+m(\cos\theta)}}{(n+2m)!} t^n.$$

A cet effet, il forme l'équation différentielle que vérifie le second membre comme fonction de t. La solution de cette équation, ramenée à celle de Bessel, est ensuite choisie en étudiant ses propriétés au voisinage du point $t=0$.

Cette formule se trouve dans une publication antérieure [L. Robin, *Fonctions sphériques de Legendre et fonctions sphéroïdales*. Tome II, Gauthier-Villars, Paris, 1958; MR **21** #737; p. 321, form. (142), ou une autre démonstration est donnée]. Elle est valable quels que soient t et θ. Elle a été aussi généralisée au cas des polynômes de Gegenbauer.
L. Robin (Paris)

6903:
Meligy, A. S. Expansions of Whittaker's function. Quart. J. Math. Oxford Ser. (2) **10** (1959), 202–205.

Verf. geht von einer bekannten Konturintegraldarstellung der confluenten hypergeometrischen Funktion aus, setzt eine Reihenentwicklung für einen der Faktoren des Integranden an und findet daraus unmittelbar eine Reihendarstellung für eine confluente hypergeometrische Funktion als unendliche Summe solcher Funktionen mit explizit bekannten Koeffizienten. Die Funktionen unter dem Summenzeichen lassen sich je als unendliche Summe Besselscher Funktionen erster Art darstellen. Hieraus folgt dann eine Darstellung der Ausgangsfunktion als unendliche Summe Besselscher Funktionen. Diese Formeln werden auf die reguläre radiale Coulombsche Wellenfunktion angewandt.
M. J. O. Strutt (Zürich)

6904:
Carlitz, L. Note on a formula of Rainville. Bull. Calcutta Math. Soc. **51** (1959), 132–133.

Extension aux polynomes de Gegenbauer d'une formule relative aux polynomes de Legendre établie par Rainville en 1945 [Bull. Amer. Math. Soc. **51** (1945), 268–271; MR **6**, 211]:

$$\left(\frac{\sin\beta}{\sin\alpha}\right)^n P_n(\cos\alpha) = \sum_{r=0}^{n} \binom{n}{r} \left[\frac{\sin(\beta-\alpha)}{\sin\alpha}\right]^r P_{n-r}(\cos\beta).$$

R. Campbell (Caen)

6905:
Aczél, J. Sur l'équation différentielle des polynomes orthogonaux classiques. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 27–29.

Reference is made to a paper by Á. Császár [same Ann. **1** (1958), 33–39; MR **21** #155], and the following is proved. Let $L(x)$ be linear and $Q(x)$ quadratic, moreover $Q(x)w'(x) = L(x)w(x)$, $Q(x)\neq 0$, $p_n(x) = [w(x)Q(x)^n]^{(n)}/w(x)$, $w(x)\neq 0$. Then the following differential equation holds:

$$Q(x)p_n''(x) + [L(x)+Q'(x)]p_n'(x) + c_n p_n(x) = 0;$$

$$c_n = -nL'(x) - \binom{n+1}{2} Q''(x).$$

G. Szegö (Stanford, Calif.)

ORDINARY DIFFERENTIAL EQUATIONS

6906:
Kolodziej, W. On an application of a certain inequality in the theory of differential equations. Wiadom. Mat. (2) **3**, 45–47 (1959). (Polish)

Let $y_1(x)$, $y_2(x)$ $(a \leq x \leq b)$ be solutions of the differential equations $y' = f(x, y)$, $y' = f(x, y) + \varphi(x, y)$ respectively, satisfying the initial conditions: $y_1(x_1) = y_1$, $y_2(x_2) = y_2$. It is assumed that the functions $f(x, y)$, $\varphi(x, y)$ are continuous and $|f(x, y)| \leq M$, $|\varphi(x, y)| \leq \mu$, $|f(x, y) - f(x, \bar{y})| \leq L|y-\bar{y}|$. The author first proves the inequality

$$|y_1(x) - y_2(x)| \leq (|y_1 - y_2| + M|x_1 - x_2| + \mu(b-a))e^{L(b-a)} \quad (a \leq x \leq b),$$

and then deduces from it the standard theorem on the dependence of the solutions on the initial conditions and parameters. For similar considerations see E. Kamke, *Differentialgleichungen reeller Funktionen* [Chelsea, New York, 1947; MR **8**, 514].
Z. Opial (Kraków)

6907:
Valat, Jean. Les solutions algébriques des systèmes d'équations différentielles du deuxième ordre, couplées non linéairement par des polynomes à coefficients constants. C. R. Acad. Sci. Paris **251** (1960), 198–200.

Pour les systèmes $x'' = X(x, y)$, $y'' = Y(x, y)$, où X et Y sont des polynomes en x et y du troisième degré au plus, l'auteur propose une méthode pour trouver les mouvements dont la trajectoire est algébrique. Ces mouvements n'existent que pour certains systèmes de la forme envisagée. En outre, les conditions initiales ne sont pas arbitraire. On utilise la théorie des fonctions complexes (elliptiques ou fuchsiennes) pour obtenir les équations paramétriques du mouvement. Cette méthode, qui permet de transformer le système différentiel en un système d'équations algébriques non linéaires, serait applicable (selon l'auteur) à d'autres systèmes différentiels.
C. Corduneanu (Iaşi)

6908:

Valat, Jean. Sur certaines solutions non algébriques des systèmes d'équations différentielles du deuxième ordre, couplées non linéairement par des polynomes à coefficients constants. C. R. Acad. Sci. Paris **251** (1960), 840–842.

L'auteur étudie les systèmes différentiels $x''=X(x, y)$, $y''=Y(x, y)$, où X et Y sont des polynomes à coefficients constants du troisième degré au plus, à l'aide des transformations laissant invariant un tel système. Par cette voie, il est possible de trouver la solution générale pour certains systèmes du type envisagé, en utilisant les fonctions elliptiques. Les solutions algébriques correspondent au cas où x et y s'expriment par des fonctions de la mêmes période, tandis que les solutions non algébriques correspondent au cas où x et y s'expriment par des fonctions dont les périodes fondamentales sont incommensurables. Un exemple est traité en détail. *C. Corduneanu* (Iași)

6909:

Salinas, Baltasar R. Sulla stabilità delle soluzioni per l'equazione differenziale del secondo ordine a coefficienti periodici. Rend. Circ. Mat. Palermo (2) **8** (1959), 206–224. (English summary)

L'Autore estende all'equazione (1) $\ddot{x}+p(t)\dot{x}+q(t)x=0$ ($p(t)$, $q(t)$ reali, continue, periodiche) noti risultati di A. Liapunov, G. Borg, E. R. van Kampen, A. Wintner (cfr. ad es. A. Wintner, Quart. Appl. Math. **16** (1958), 175–178 [MR **20** #1815]) relativi al caso $p(t)\equiv 0$. Se ω è il periodo di p, q e si pone $P(t)=\int_0^t \exp(-\int_0^s p(u)du)ds$, il risultato principale è il seguente: se $\int_0^\omega p(t)dt \geqq 0$, $q(t)\not\equiv 0$,

$$\int_0^\omega [P(\omega)-P(t)+P'(\omega)P(t)][P'(t)]^{-1}q(t)\,dt \geqq 0,$$

$$\int_0^\omega [P(\omega)-P(t)+P'(\omega)P(t)][P'(t)]^{-1}|q(t)|\,dt \leqq [1+P'(\omega)]^{1/2},$$

ogni soluzione $x(t)$ della (1) è limitata insieme alla derivata $\dot{x}(t)$ per $t\to+\infty$, ed inoltre i due fattori caratteristici associati alla (1) sono immaginari o positivi di modulo <1. L'Autore mostra anche che le condizioni suddette sono, in un certo senso, le migliori possibili. *R. Conti* (Florence)

6910:

Driscoll, Richard J. Existence theorems for certain classes of two-point boundary problems by variational methods. Pacific J. Math. **10** (1960), 91–115.

Se $f(x, y, z)$ è un vettore a n componenti funzioni della variabile reale x e dei vettori y, z a n componenti ciascuno, sotto opportune ipotesi su $f(x, y, z)$ l'A. dimostra che l'integrale $\int_a^b (|y'-z|^2+|z'-f(x, y, z)|^2)dx$ considerato nella classe dei vettori y, z assolutamente continui in $[a, b]$, $|y'(x)|^2$, $|z'(x)|^2$ integrabili in $[a, b]$ e tali che $y(a)=y_1$, $y(b)=y_2$ ha per minimo lo zero.

Sempre con metodi variazionali l'A., considerando l'integrale $\int_0^\infty (|y'|^2+2g(x, y))dx$ ed estendendo un teorema particolare per la così detta equazione di Thomas-Fermi, $y''=x^{-1/2}y^{3/2}$, dimostra un teorema generale di esistenza per un sistema della forma $y''=g(x, y)$, $0<x<\infty$, $y(0)=y_0$, $y(x)$ limitata in $[0, \infty)$. *G. Sansone* (Florence)

6911:

Dikiĭ, L. A. On boundary conditions depending on an eigenvalue. Uspehi Mat. Nauk **15** (1960), no. 1 (91), 195–198. (Russian)

Folgendes Randwertproblem (1) sei gegeben: $-u''+U(x)u=\lambda u$, $u'(0)-N(\lambda)u(0)=u(a)=0$, wobei $U(x)$ stetig ist in $[0, a]$ und $N(\lambda)$ eine gegebene Funktion des Eigenwertes λ ist. Ferner sei $\varphi(x, \lambda)$ eine Lösung der Gleichung mit $\varphi(a, \lambda)=0$. Dann ist $M(\lambda)=\varphi'(0, \lambda)/\varphi(0, \lambda)$ eindeutig bestimmt für alle λ, welche den Nenner nicht zu Null machen. Diejenigen λ, für die $M=N$, stellen die Eigenwerte unseres Problems dar. Die folgenden Sätze werden bewiesen: (1) Wenn auf jedem Kurvenzweig von $M(\lambda)$ je ein Punkt genommen wird, und $\{\lambda_i^*\}$ die Gesamtheit der Abzissen dieser Punkte ist, dann ist das System der Funktionen $\varphi(x, \lambda_i^*)$ vollständig im Raum $\mathscr{L}^2(0, a)$. (2) Wenn die Funktion $N(\lambda)$ stetig ist, und $\limsup_{\lambda\to-\infty} N(\lambda)/(-\lambda)^{1/2} > -1$, dann ist das System der Eigenfunktionen des Problems (1) vollständig. *G. Goes* (Evanston, Ill.)

6912:

Atkinson, F. V. A constant area principle for steady oscillations. I. J. Math. Anal. Appl. **1** (1960), 133–144.

The author considers the system of two differential equations

$$\text{(1)}\qquad \dot{u} = -\partial H(u, v, t)/\partial p, \quad \dot{v} = \partial H(u, v, t)/\partial u$$

and observes that the condition that for some solution $u(t)$, $v(t)$ of (1) $H(u(t), v(t), t)$ is approximately constant is, in general, insufficient to yield suitable information about its behavior. In order to obtain a different approximate first integral of (1), sometimes more convenient, he replaces this condition by the assertion that the area enclosed by the corresponding level curve of H is approximately constant.

For a fixed solution $u(t)$, $v(t)$ of (1) consider the locus in the (ξ, η)-plane defined by $H(u(t), v(t), t)=H(\xi, \eta, t)$ and denote by $A(t)$ the area enclosed by this curve. The author first estimates the range of change of $A(t)$ and then, using these estimates, determines the limiting form of the solutions of the system

$$\text{(2)}\qquad \dot{u} = -\varepsilon\partial H/\partial v, \quad \dot{v} = \varepsilon\partial H/\partial u$$

when $\varepsilon\to 0$. He proves the following theorem: For the function $H(\xi, \eta, t)$ let there be a family of curves $C_1(t)$ $(0\leqq t\leqq t_0)$, such that (i) $C_1(t)$ is a closed rectifiable curve, (ii) $H(\xi, \eta, t)$ is constant for $(\xi, \eta)\in C_1(t)$ and fixed t, (iii) the area $A_1(t)$ enclosed by $C_1(t)$ is constant for $0\leqq t\leqq t_0$, (iv) $C_1(t)$ varies continuously with t, (v) there is an open region E of the Euclidean (ξ, η, t)-space, including all (ξ, η, t) such that $(\xi, \eta)\in C_1(t)$, $0\leqq t\leqq t_0$, in which H is continuously twice differentiable and $|\text{grad } H|$ is bounded away from zero, (vi) the lengths of the curves $C_1(t)$, and of all other level curves of H lying in E, lie between fixed positive limits. Then for any solution of (2) such that $\{u(0), v(0)\}\in C_1(0)$ we have, as $\varepsilon\to 0$, and for $0\leqq t\leqq t_0$, $\min_{(\xi,\eta)\in C_1(t)} [(u(t)-\xi)^2+(v(t)-\eta)^2]^{1/2}=O(\varepsilon)$.

Z. Opial (Kraków)

6913:

Sansone, Giovanni. Sopra l'equazione differenziale del secondo ordine del Dini $xy''+y'=\sin y$. Comportamento degli integrali per $x\to\infty$. Ann. Mat. Pura Appl. (4) **50** (1960), 439–465.

A penetrating study of the integrals in the real field of

the differential equation in the title. The starting point is an unpublished result of Dini concerning the holomorphic solutions of the differential equation, which the author found among Dini's manuscripts. Dini's theorem is the following. Let $F(y)$ be an entire function, real for real y, such that $|F^{(r)}(a_0)| \leq \lambda$ $(r=0, 1, 2, \cdots)$. Then $xy''+y'=F(y)$ has one and only one holomorphic solution in $|x|<2/\lambda$ which satisfies the initial condition $y(0)=a_0$. Dini's proof, reproduced in the paper, is by a method of majorants.

In the real field, there are only three types of solutions of $xy''+y'=\sin y$: (a) constant solutions $y\equiv k\pi$ (k integer); (b) non-constant non-oscillating solutions, for which $\lim_{x\to\infty} y(x)=2l\pi$ and $\lim_{x\to 0} y(x)=\pm\infty$; (c) oscillating solutions defined for $x>0$. For these the maxima tend to the limit $2l\pi+p\pi$, and the minima to $2l\pi+(2-p)\pi$, where l is an integer and $1\leq p<2$. All solutions are bounded above and below for $x\geq x_0$. If $y(x, \alpha)$ is the solution satisfying $y(x_0, \alpha)=0$, $y'(x_0, \alpha)=\tan\alpha$, then there exists an α_1 such that $\lim_{x\to\infty} y(x, \alpha_1)=2\pi$, all integrals $y(x, \alpha)$ with $\alpha<\alpha_1$ are oscillating, and all integrals with $\alpha>\alpha_1$ intersect the line $y=2\pi$. Two integrals, both contained in the strip $\pi\leq y\leq 2\pi$ and both tending to 2π as $x\to\infty$, do not intersect. If k is a positive [negative] integer, there is an increasing [decreasing] solution $y(x, \alpha_k)$ for which $\lim_{x\to\infty} y(x, \alpha_k)=2k\pi$; for $\alpha>\alpha_k$ [$\alpha<\alpha_k$], $y(x, \alpha)$ intersects the line $y=2k\pi$; and $\alpha_{k+1}>\alpha_k$ [$\alpha_{k+1}<\alpha_k$] for $k>0$. A diagram is given. *G.-C. Rota* (Cambridge, Mass.)

6914:

Moretto, Sergio. Sull'esistenza di soluzioni periodiche per l'equazione $y'=f(x, y)$. Ann. Univ. Ferrara. Sez. VII (N.S.) 8 (1958/59), 61–67. (French summary)

L'A. prova il seguente teorema. Siano $\sigma(x)$, $\tau(x)$ due funzioni periodiche, di periodo ω, continue, a variazione limitata, $\sigma(x)<\tau(x)$. Sia $f(x, y)$ una funzione reale definita nella striscia D: $|x|<+\infty$, $\sigma(x)\leq y\leq\tau(x)$ periodica in x, di periodo ω, sommabile in x su ogni sezione di D con le rette $y=\text{cost.}$, e tale che per qualche $P(x)\geq 0$, sommabile in $|x|<+\infty$, sia verificata la disuguaglianza $|f(x, \bar{y})-f(x, \bar{\bar{y}})|\leq P(x)|\bar{y}-\bar{\bar{y}}|$, per quasi tutte le x e per tutte le coppie $\bar{y}$, $\bar{\bar{y}}$, $\sigma(x)\leq\bar{y}$, $\bar{\bar{y}}\leq\tau(x)$. Infine una delle due funzioni

$$\sigma(x)-\int_0^x f(t, \sigma(t))\,dt, \quad \tau(x)-\int_0^x f(t, \tau(t))\,dt$$

sia non crescente, l'altra non decrescente. Allora l'equazione $y'=f(x, y)$ ammette almeno una soluzione, nel senso di Carathéodory, periodica, di periodo ω. In particolare viene considerato il caso di $\sigma(x)$, $\tau(x)$ costanti.

R. Conti (Florence)

6915:

Santoro, Paolo. Sulla stabilità uniforme e asintotica uniforme in prima approssimazione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 336–341.

Data l'equazione (1) $\dot{x}=A(t)x+f(t, x)$, con $A(t)$ matrice $n\times n$ continua in $-\infty\leq T<t<+\infty$, $f(t, x)$ n-colonna continua in Ω: $T<t<+\infty$, $0\leq\|x\|^2=\sum_{i=1}^n |x_i|^2$, $f(t, 0)=0$, sia $\Lambda(t)$ il più grande degli autovalori della matrice hermitiana $(A+A^*)/2$.

L'A. prova i seguenti teoremi. I. Se (2) $x^*f(t, x)\leq\|x\|\omega(t, \|x\|)$, $(t, x)\in\Omega$, con $\omega(t, u)$ continua per $T<t<+\infty$, $0\leq u$, monotona non decrescente in u per ogni t, $\omega(t, 0)=0$, e se $\int_{t_0}^t \Lambda(\tau)d\tau\leq c$, $T<t_0<t<+\infty$ per una costante $c\geq 0$ tale che la soluzione nulla dell'equazione (scalare) $\dot{u}=e^c\omega(t, u)$ sia uniformemente stabile, allora anche la soluzione nulla di (1) è uniformemente stabile. Se $A\equiv 0$ si ha $\Lambda\equiv 0$ e quindi si può prendere $c=0$. II. Se vale (2) e se $\int_{t_0}^t (\Lambda(\tau)+\nu)d\tau\leq N$, $T<t_0<t<+\infty$ per due costanti $N\geq 0$, $\nu>0$ tali che la soluzione nulla dell'equazione (scalare) $\dot{u}=-\nu u+e^N\omega(t, u)$ sia asintoticamente uniformemente stabile, allora anche la soluzione nulla di (1) è asintoticamente uniformemente stabile.

Questi risultati, ottenuti mediante maggiorazioni elementari di $\|x(t)\|$, intersecano quelli, recentemente ottenuti da A. Stokes [Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 231–235; MR **21** #2768] mediante metodi dell'analisi funzionale; le ipotesi di Stokes sono più larghe per quanto riguarda la $A(t)$, più restrittive per quanto riguarda la $f(t, x)$.

{Alcuni errori di stampa, facilmente individuabili.}

R. Conti (Florence)

6916:

Matuda, Tizuko. Sur les points singuliers des équations différentielles ordinaires du premier ordre. VI. Nat. Sci. Rep. Ochanomizu Univ. **10** (1959), 7–18.

Let $P(x, y)$ and $Q(x, y)$ be two polynomials in y without common factors, and with coefficients holomorphic in x for $|x|<\delta$. Assume that $P(0, 0)\neq 0$ and $Q(0, 0)\neq 0$. Let ν be a positive integer. In this paper, assuming the existence of a periodic solution of the equation

$$dy/dt = P(0, y)/y^\nu Q(0, y) \quad (0\leq \operatorname{Im} t\leq\rho),$$

such that its orbit passes the singular point $y=0$, the author studies the behavior of solutions of the equation

$$x\,dy/dx = P(x, y)/y^\nu Q(x, y)$$

as x tends to 0 on the positive part of real axis. For the most part the results are similar to those obtained previously by the author for the case $\nu=1$. [Same Rep. **5** (1954/55), 1–4, 175–177; **8** (1957), 1–6; MR **16**, 1023; **17**, 847; **20** #3319. See also ibid. **2** (1951), 13–17; **4** (1953), 36–39; Sûgaku **8** (1956/57), 139–148; **14**, 274; **15**, 126; **21** #1435.]

Y. Sibuya (New York)

6917:

Tondl, Aleš. Méthode pour la détermination des zones labiles dans les systèmes quasi-harmoniques. Apl. Mat. **4** (1959), 278–289. (Czech. Russian and French summaries)

On considère des systèmes paramétriques quasi-harmoniques dont le mouvement peut être décrit par des équations différentielles de la forme

$$\ddot{y}_s+\sum_{k=1}^n a_{sk}y_k+\varepsilon\sum_{k=1}^n (q_{sk}\dot{y}_k+p_{sk}y_k)=0 \quad (s=1, 2, \cdots, n),$$

où a_{sk} sont des constantes et $q_{sk}=q_{sk}(t)$, $p_{sk}=p_{sk}(t)$ sont des fonctions périodiques continues, de période $2\pi/\omega$, développables en séries de Fourier, et ε est un petit paramètre. L'A., tout en utilisant les résultats obtenus récemment par M. G. Kreĭn [*Pamyati Aleksandra Aleksandroviča Andronova*, pp. 413–498, Izdat. Akad. Nauk SSSR, Moscow, 1955; MR **17**, 738] et S. N. Šimanov [cf. Prikl. Mat. Meh. **16** (1952), 129–146; MR **13**, 745], expose une méthode de détermination des zones labiles des systèmes ci-dessus. On examine ensuite les intervalles des solutions instables pour des ω qui se trouvent en proximité des

valeurs $\omega_0=(\Omega_k\pm\Omega_j)/N$ $(k, j=1, 2, \cdots, n)$, $N>0$ entier, où Ω_k, Ω_j sont les fréquences du système obtenu pour $\varepsilon=0$.

L'article se conclut par l'application des résultats obtenus au système suivant à deux degrés de liberté $\ddot{x}+\Omega_1{}^2x=\varepsilon(-2\delta_1\dot{x}+q_1x\cos\omega t+q_2y\cos\omega t)$, $\ddot{y}+\Omega_2{}^2y=\varepsilon(-2\delta_2\dot{y}+Q_2y\cos\omega t+Q_1x\cos\omega t)$, où $\Omega_2>\Omega_1>0$; $\delta_1>0$, $\delta_2>0$; et Ω_1/Ω_2 n'est pas un nombre rationnel.

D. Mangeron (Iași)

6918:

Halanay, A. Solutions presque-périodiques des systèmes d'équations différentielles à argument retardé contenant un petit paramètre. Com. Acad. R. P. Romîne **9** (1959), 1237–1242. (Romanian. Russian and French summaries)

Considérons le système avec retardement τ et petit paramètre ε $\dot{h}(t)=\varepsilon H_1h(t)+\varepsilon H_2h(t-\varepsilon\tau)+\varepsilon B_1[h(t), h(t-\varepsilon\tau)]+\varepsilon\Gamma(t, h(t), h(t-\varepsilon\tau), \varepsilon)$ avec certaines conditions sur les matrices H_1, H_2, B_1, Γ (par exemple, H_1+H_2 a les valeurs propres avec partie réelle négative, et Γ est presque-périodique en t, uniformément par rapport aux autres variables). Alors, par une méthode d'approximation successive, et en utilisant essentiellement des résultats d'un travail précédent de l'auteur, on démontre que pour $0<\varepsilon<\varepsilon_0$, le système a une solution unique presque-périodique $h(t)$ qui pour $\varepsilon\to 0$ tend vers 0 aussi.

Avec certains changements de variables, des systèmes plus généraux sont réduits au cas envisagé.

{Selon l'avis du rapporteur, un travail faisant synthèse des résultats importants obtenus par Halanay dans la théorie qualitative des équations différentielles avec retardement serait très utile.} *S. Zaidman* (Bucharest)

PARTIAL DIFFERENTIAL EQUATIONS

See also 6897.

6919:

Goldhagen, E. Sur les équations aux dérivées partielles du Ier ordre à deux fonctions inconnues de p variables indépendantes, qui admettent des intégrales générales explicites. Acad. R. P. Romîne. Fil. Iași. Stud. Cerc. Şti. Mat. **10** (1959), 35–71. (Romanian. Russian and French summaries)

L'A., continuando le sue ricerche concernenti il problema d'integrazione esplicita di vari equazioni alle derivate parziali oppure di sistemi di tali equazioni [Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 623–644; Acad. R. P. Romîne. Fil. Iași. Stud. Cerc. Şti. Mat. **7** (1956), no. 1, 51–70; **8** (1957), no. 1, 75–106; MR **17**, 490; **20** #1104], studia in questa sua le classi di equazioni alle derivate parziali del primo ordine con due funzioni incognite in p variabili indipendenti

$$\text{(a)}\qquad \Phi\left(x_i, z_1, z_2, \frac{\partial z_1}{\partial x_i}, \frac{\partial z_2}{\partial x_i}\right)=0 \quad (i=1, 2, \cdots, p),$$

le quali ammettono un integrale generale di forma

$$\text{(b)}\qquad z_1=f(x_i, \zeta_{i_1i_2\cdots i_p}),\quad z_2=g(x_i, \zeta_{i_1i_2\cdots i_p})$$
$$(i_1+i_2+\cdots+i_p=0, 1, \cdots, n),$$

essendovi f e g funzioni date continue, con derivate parziali continue sino all'ordine due incluso, ζ è una funzione qualsiasi di $x_1, x_2, \cdots, x_p$ e

$$\zeta_{i_1i_2\cdots i_p}=\frac{\partial^{i_1+i_2+\cdots+i_p}\zeta}{\partial x_1{}^{i_1}\partial x_2{}^{i_2}\cdots\partial x_p{}^{i_p}},\quad \zeta_{0\cdots 0}=\zeta.$$

Si dimostra che l'equazione (a) ammetente l'integrale generale esplicito (b) investe la forma

$$\text{(c)}\qquad \Phi\left(x_i, z_1, z_2, \frac{\partial z_1}{\partial x_i}\right)=0,$$

oppure risulta quasilineare, mentre il suo integrale generale ne ha la forma $z_1=\zeta$, $z_2=F(x_i, \zeta, \partial\zeta/\partial x_i)$. Si studiono poscià le equazioni quasilineari $\partial z_1/\partial x_1-\partial z_2/\partial x_2=z_1\partial z_2/\partial x_3-z_2\partial z_1/\partial x_3$, per cui si perviene al risultato che la soluzione esplicita corrispondente ne è tuttora anche il suo integrale generale. *D. Mangeron* (Iași)

6920:

Haimovici, Mendel. Alcune proprietà delle decomposizioni dei sistemi differenziali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 646–652.

Dans une note précédente, l'auteur a défini la décomposition régulière d'un système différentiel extérieur. Si $S=S^{(1)}+S^{(2)}$ est une telle décomposition, il a établi que l'intégrale générale s'obtient en intégrant d'abord $S^{(1)}$ et ensuite $S^{(2)}$ sur une variété intégrale générique de $S^{(1)}$. Il donne ici une classification des éléments plans intégraux singuliers d'un système différentiel extérieur, et il étudie leurs propriétés en relation avec la décomposition de S en $S^{(1)}$ et $S^{(2)}$. *G. Papy* (Brussels)

6921:

Haimovici, M. Sur les prolongements partiels des systèmes différentiels extérieurs. Acad. R. P. Romîne. Fil. Iași Stud. Cerc. Şti. Mat. **9** (1958), no. 2, 199–221. (Romanian. Russian and French summaries)

L'auteur expose quelques résultats concernant les prolongements (partiels ou totaux) des systèmes différentiels extérieurs. Dans le chapitre III, il étudie un prolongement particulier qu'il appelle un prolongement linéaire régulier. Il donne des conditions nécessaires et suffisantes pour qu'un système d'équations extérieures donné possède des prolongements privilégiés. *G. Papy* (Brussels)

6922:

Dezin, A. A. Existence and uniqueness theorems for solutions of boundary problems for partial differential equations in function spaces. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 21–73. (Russian)

Le but de cet article est d'illustrer, sur des exemples simples, les méthodes de S. L. Sobolev, K. O. Friedrichs, M. I. Višik, J. L. Lions, etc.

L'auteur montre que ces méthodes ont pour modèle l'équation linéaire $Au=f$ dans les espaces à un nombre fini de dimensions, dont toute la discussion repose sur la considération de l'équation homogène et de l'équation adjointe. Il décrit les espaces fonctionnels (généralement, des espaces hilbertiens) dans lesquels les opérateurs aux dérivées partielles de la physique mathématique (les plus simples) se laissent encadrer dans ce schéma, et en tire les théorèmes d'existence et unicité. Naturellement, on

emploie les définitions généralisées des opérateurs considérés et aussi les solutions généralisées au sens de S. L. Sobolev. *G. Marinescu* (Bucharest)

6923:

Methée, P. D. ★Transformées de Fourier de distributions invariantes liées à la résolution de l'équation des ondes. La théorie des équations aux dérivées partielles. Nancy, 9–15 avril 1956, pp. 145–163. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXI. Centre National de la Recherche Scientifique, Paris, 1956. 187 pp. 1500 francs.

Dans sa thèse [Comment. Math. Helv. **28** (1954), 225–269; MR **16**, 225] l'auteur a déterminé l'espace de toutes les distributions solutions invariantes par le groupe des rotations propres de Lorentz de l'équation des ondes $(\square+\kappa)T=0$ sur R^n $(n\geqq 3$, κ complexe, $\square=\partial^2/\partial x_n^2-\partial^2/\partial x_1^2-\cdots-\partial^2/\partial x_{n-1}^2)$, en donnant une base de 3 distributions. Suivant que κ est non réelle, négative, positive ou nulle, le sous-espace des solutions invariantes tempérées est à 0, 1, 2 ou 3 dimensions (comme il en est de même avec celui de l'équation $(u-\kappa)T=0$, où $u=x_n^2-x_1^2-\cdots-x_{n-1}^2$); les transformées de Fourier d'une base sont calculées, ainsi que celles de certaines distributions (par exemple, les H_k et $\bar{H}_k$ qui engendrent avec les $\square^k\delta$ l'espace des distributions invariantes portées par le cône $u=0$) liées aux solutions. En particulière, on retrouve et justifie des formules avec des "fonctions singulièrers" $(\Delta, \Delta^{(1)}, \bar{\Delta}, \varepsilon)$ de l'éléctrodynamique quantique.

Les résultats ont été annoncés dans deux notes précedentes [C. R. Acad. Sci. Paris **240** (1955), 1179–1181; **241** (1955), 684–686; MR **16**, 1101; **17**, 845].

S. Łojasiewicz (Kraków)

6924:

Shanahan, John P. On uniqueness questions for hyperbolic differential equations. Pacific J. Math. **10** (1960), 677–688.

Le problème aux valeurs initiales

$$\frac{\partial^2 z}{\partial x\partial y}=f(x,y,z,p,q)\quad\left(p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}\right),$$

$$z(x,0)=\sigma(x),\quad z(0,y)=\tau(y),\quad \text{avec } \sigma(0)=\tau(0)=z_0,$$

posé dans un rectangle $R=[0,a]\times[0,b]$, possède une et une seule solution, sous des hypothèses convenables sur σ, τ et f, dont la plus restrictive paraît être que, pour tout $(x,y)\in R$, on ait

$$|f(x,y,z,p,q)-f(x,y,z',p',q')|\leqq \varphi(x,y,|z-z'|,|p-p'|,|q-q'|),$$

où $\varphi(x,y,z,p,q)$ désigne une fonction continue de x, y, non décroissante par rapport à z, p, q séparément et telle que la seule solution de

$$z(x,y)=\int_0^x\int_0^y \varphi[s,t,z(s,t),\frac{\partial z}{\partial x}(s,t),\frac{\partial z}{\partial y}(s,t)]\,ds\,dt$$

dans $[0,\alpha]\times[0,\beta]$ soit $z=0$, et ce, pour tout $(\alpha,\beta)\in\,]0,a]\times]0,b]$.

Cette solution est la limite uniforme des approximations successives définies par $z_0(x,y)=\sigma(x)+\tau(y)-z_0$,

$$z_n(x,y)=z_0(x,y)+\int_0^x\int_0^y f[x,y,z_{n-1}(s,t),\frac{\partial z_{n-1}}{\partial x}(s,t),\frac{\partial z_{n-1}}{\partial y}(s,t)]\,ds\,dt.$$

L'auteur établit ce théorème et le complète par un théorème d'unicité suggéré par le théorème de Nagumo pour les équations différentielles ordinaires [cf. E. Kamke, *Differentialgleichungen reeller Funktionen*, Akademische Verlag., Leipzig, 1930; p. 97]. *H. G. Garnir* (Liège)

6925:

Zerner, Martin. Solutions de l'équation des ondes présentant des singularités sur une droite. C. R. Acad. Sci. Paris **250** (1960), 2980–2982.

L'A. montre la proposition suivante: Étant donnés L un opérateur différentiel du second ordre à coefficients constants hyperbolique ou ultra-hyperbolique, D une bicaractéristique de L, il existe alors une distribution u, solution de $Lu=0$, qui est une fonction indéfiniment différentiable en dehors de D, mais qui ne l'est pas au voisinage de D. Après avoir montré ceci pour l'équation hyperbolique, il montre une expression de telle u pour le cas général que voici: On se ramène à la situation suivante: $(s,t,z_1,\cdots,z_{n-2})=(s,t,z)$ est un système de coordonnées dans lequel D a pour équations $t=z=0$ et L a comme expression $L=\partial^2/\partial s\partial t-M(\partial/\partial z)$, où M est un polynome du second degré. Soit $\psi(t)\in\mathcal{D}_t$, à support contenu dans $t\geqq 0$. Alors

$$u(s,t,z)=\int_{R_\zeta^{n-2}}\int_{-\infty}^{\infty}\psi\left(\frac{|\zeta|^2}{\tau}\right)\exp\left[\frac{M(i\zeta)}{i\tau}s\right]\exp(i\tau t+i\zeta z)\,d\zeta\,d\tau$$

répond à notre demande. *S. Mizohata* (New York)

6926:

Szarski, J.; Szmydt, Z.; Ważewski, T. Remarque sur la régularité des intégrales des équations différentielles hyperboliques du second ordre. Ann. Polon. Math. **6** (1959), 241–244.

In seguito alla Nota del H. Schaefer [cf. Jber. Deutsch. Math. Verein. **58** (1955), Abt. 1, 39–42; MR **17**, 624], il quale ha dato una dimostrazione del fatto che l'equazione

$$(1)\qquad u_{xy}(x,y)=f(x,y,u(x,y),u_x(x,y),u_y(x,y)),$$

ove $f(x,y,u,p,q)$ è una funzione di classe C^1 nel parallelipipedo $|x|$, $|y|<d$, $|u|$, $|p|$, $|q|<M$ $(d>0,\ M>0)$, ammette nel rettangolo sufficientemente piccolo $|x|$, $|y|<d_1$ $(d_1\leqq d)$ una soluzione unica di classe C^2 che si annulla lungo le caratteristiche $x=0$ e $y=0$, gli AA. ne danno nella presente un'altra dimostrazione più semplice, basata sui noti teoremi concernenti la derivazione degli integrali delle equazioni differenziali ordinarie dipendenti da un parametro.

Il teorema 2 dell'articolo stesso che ne collega in certi ipotesi l'esistenza din una soluzione di classe C^{n+1} $(n\geqq 1)$ per (1) allorchè $f(x,y,u,p,q)$ vi è una funzione di classe C^n, permette di ottenere i teoremi che ne assicurano l'esistenza e l'unicità nel caso del problema di Darboux spettante all'equazione (1) della soluzione di classe C^n $(n\geqq 2)$ se vi si utilizzano i risultati noti concernenti l'esistenza e l'unicità delle soluzioni di classe C^1 per tale problema. *D. Mangeron* (Iași)

6927:

Amerio, Luigi. Quasi-periodicità degli integrali ad energia limitata dell'equazione delle onde, con termine noto quasi-periodico. I, II, III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 147–152, 322–327, 461–466.

Ce travail est une contribution essentielle au problème de la presque-périodicité des solutions de l'équation des ondes nonhomogène, avec partie droite presque-périodique.

Soit Ω un domaine borné de l'espace S_m, $x=(x_1, \cdots, x_m)$ et $J \equiv -\infty < t < +\infty$. On considère des solutions faibles de l'équation des ondes

$$(1) \qquad u_{tt}(x, t) = \sum_{j,k=1}^{m} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u}{\partial x_k}\right) - a(x)u + f(x, t)$$

avec la condition $u|_S = 0$, $S = \operatorname{Fr}\Omega$, $f(x, t)$ étant presque-périodique de J à $L^2(\Omega)$. Soit D^0 l'espace de Friedrichs obtenu des fonctions $v(x) \in C^1(\bar{\Omega})$, nulles dans le voisinage de S, par complétion dans la métrique

$$\|v(x)\|^{\,2}_{D^0} = \left\{\int_\Omega \Big(\sum_{j,k=1}^{m} a_{jk}(x)\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_k} + a(x)v^2\Big)\, d\Omega\right\}^{1/2},$$

et soit $E = D^0 \times L^2(\Omega)$, l'espace de l'énérgie.

Le rapporteur avait démontré (en 1958), que toute solution $u(x, t)$ de (1) qui engendre un vecteur $\{u(x, t), u_t(x, t)\}$ avec trajectoire relativement compacte de $t \in J$ à E, est presque-périodique de J à E. Maintenant Amerio établit que si le vecteur $\{u(x, t), u_t(x, t)\}$ est seulement borné de J à E, il est aussi presque-périodique. Dans la démonstration on utilise un lemme important sur la convergence uniforme des suites monotones de fonctions presque-périodiques (Note II), rappelant le lemme bien-connu de Dini sur la convergence uniforme des suites monotones de fonctions continues. *S. Zaidman* (Bucharest)

6928:

Li, Dè-Yuan'. Uniqueness of Cauchy's problem for a parabolic type of equation. Dokl. Akad. Nauk SSSR **129** (1959), 979–982. (Russian)

The author investigates Cauchy's problem for the parabolic equation

$$(1) \qquad L^*u \equiv \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ik}(x_1, \cdots, x_n, t)\frac{\partial u}{\partial x_k}\right] + \sum_{i=1}^{n} b_i(x_1, \cdots, x_n, t)\frac{\partial u}{\partial x_i} + c(x_1, \cdots, x_n, t) - \frac{\partial u}{\partial t} = d(x_1, \cdots, x_n, t),$$

defined in a cylinder $G\{(x_1, \cdots, x_n) \in D,\ 0 \leqq t \leqq T\}$, where D is an n-dimensional region. He supposes that a_{ik} have three continuous derivatives with respect to x_i and one continuous derivative with respect to t; the operator

$$L_1^* = \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ik}\frac{\partial}{\partial x_k}\right]$$

is uniformly elliptic, i.e., two constants $a_0 > 0$, $a_1 > 0$ exist such that for arbitrary real $\xi_1, \cdots, \xi_n$ and for any point $P \in G$

$$a_0 \sum_{i=1}^{n} \xi_i^2 \leqq \sum_{i,k=1}^{n} a_{ik}(x_1, \cdots, x_n, t)\xi_i\xi_k \leqq a_1 \sum_{i=1}^{n} \xi_i^2.$$

Functions b_i, c, d are supposed to be bounded.

Let $\Pi \subset D$ be an $(n-1)$-dimensional hypersurface with continuous normal, let Γ be the n-dimensional cylinder $\Gamma\{(x_1, \cdots, x_n) \in \Pi,\ 0 \leqq t \leqq T\}$. Then the following theorem holds.

The solution u of the Cauchy problem

$$(2) \qquad u(x_1, \cdots, x_n, t)|_\Gamma = f(x_1, \cdots, x_n, t), \qquad \frac{\partial u}{\partial \eta}(x_1, \cdots, x_n, t)|_\Gamma = g(x_1, \cdots, x_n, t)$$

(η is the normal to Γ) for the equation (1) is unique in the class K of functions with continuous second-order derivatives with respect to x_i and with continuous derivative with respect to t.

The proof is based on an inequality of Heinz's type, derived in the paper for functions of the so-called class F, and on two auxiliary theorems. Complete proofs are not presented (as usual in the Dokl.) and will be published later.

Finally, the author presents a uniqueness theorem for the homogeneous Cauchy problem for the quasilinear equation

$$\sum_{i,k=1}^{n} a_{ik}\left(x_1, \cdots, x_n, t, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)\frac{\partial^2 u}{\partial x_i \partial x_k} - \frac{\partial u}{\partial t} = h\left(x_1, \cdots, x_n, t, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right).$$

K. Rektorys (Prague)

6929:

Friedman, Avner. On quasi-linear parabolic equations of the second order. II. J. Math. Mech. **9** (1960), 539–556.

This paper supplements an earlier work of the author in same J. **7** (1958), 793–809 [MR **22** #140], in which existence and uniqueness theorems were proved for the first mixed boundary-value problem for a quasi-linear parabolic equation of second order. The existence proof in that paper was incomplete, since it hinged on an estimate which was derived using results not yet established. In the present paper, the author supplies a new derivation of the estimate under weaker conditions, using the fundamental solution constructed by W. Pogorzelski [Ricerche Mat. **5** (1956), 25–57; Ann. Polon. Math. **4** (1957), 61–92; MR **18**, 47; **21** #3656a]. This leads to a corresponding weakening of the conditions in the original existence theorem. *J. Elliott* (New York)

6930:

Mlak, W. Parabolic differential inequalities and Chaplighin's method. Ann. Polon. Math. **8** (1960), 139–153.

Consider the boundary value problem (1) $\partial u/\partial t = \partial^2 u/\partial x^2 + f(x, t, u)$ for $a \leqq x \leqq b$, $0 \leqq t \leqq T$; (2) $u(x, t) = \varphi(x, t)$ for $(x, 0)$, $a \leqq x \leqq b$, and for (a, t) and (b, t), $0 \leqq t \leqq T$. $f(x, t, u)$, $\varphi(x, t)$ are assumed to be continuous in all their variables and $f(x, t, u)$ is also locally Lipschitz continuous in (x, u). In the first part of the paper the author proves that there exist "maximal" and "minimal" solutions of (1), (2) in the family of all solutions. However, from the reviewer's paper [see preceding review] it follows that the system (1), (2) has only one solution! In the second part of the paper the author derives some comparison theorems by the method of Westphal. The results are too involved to be stated here. *A. Friedman* (Minneapolis, Minn.)

6931:

Robinson, Stewart M. Some properties of the fundamental solution of the parabolic equation. Duke Math. J. **27** (1960), 195–220.

Let the coefficients $a_{ij}=a_{ji}$, a_i and a $(i,j=1,2,\cdots,n)$ of

$$L(u) = a_{ij}(x_1,\cdots,x_n,t)\frac{\partial^2 u}{\partial x_i\partial x_j}+a_i(x_1,\cdots,x_n,t)\frac{\partial u}{\partial x_i}+a(x_1,\cdots,x_n,t)u-\frac{\partial u}{\partial t}$$

be bounded and continuous in (x,t) and satisfy a uniform Hölder condition with respect to the space variables $x_1,\cdots,x_n$, which range over a not necessarily bounded domain Ω of n-space. Assume further that the operator $L(u)+\partial u/\partial t$ is uniformly elliptic throughout the domain of definition of a_{ij}. Under these restrictions, the author constructs in Ω a fundamental solution of $L(u)=0$. For this purpose, he starts with a parametrix

$$\begin{aligned} W^M(A,t,B,s) &= (2\sqrt{\pi})^{-n}(\det B_{ij}(M,t,s))^{1/2} \\ &\quad\times\exp(-B_{ij}(M,t,s)(x_i-y_i)(x_j-y_j)), \quad t>s, \\ &= 0, \quad t\leq s. \end{aligned}$$

Here $A=(x_1,\cdots,x_n)$, $B=(y_1,\cdots,y_n)$ and $M=(u_1,\cdots,u_n)$ are points in Ω, and the matrix $(B_{ij}(A,t,s))=(b_{ij}(A,t,s))^{-1}$, where $b_{ij}(A,t,s)=\int_s^t a_{ij}(A,v)dv$, $0\leq s<t$. The above parametrix satisfies the equation $a_{ij}(M,t)\times W^M_{x_ix_j}(A,t,B,s)=W_t{}^M(A,t,B,s)$ for $t\neq s$. The author's construction may be compared with that due to D. G. Aronson [Illinois J. Math. **3** (1959), 580–619; MR **21** #6480], who started with a different parametrix.

K. Yosida (Tokyo)

6932:

Bass, G. I.; Kostyučenko, A. G. The principle of limit amplitude. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 153–163. (Russian)

Le problème se pose classiquement de la manière suivante: On cherche des classes d'unicité pour les solutions dans tout l'espace de l'équation de Helmholtz: $\Delta u+\omega^2 u=f(x)$, $x=(x_1,x_2,x_3)$, $f(x)$ étant assez régulière et à support compact. Peut-être mieux serait de considérer directement (a) le principe de l'amplitude limite et (b) le principe de l'absorption limite comme deux questions de comportement asymptotique (a) des solutions $V(x,t)$ telles que $V(x,0)=V_t(x,0)=0$ de l'équation

(1) $\quad V_{tt}(x,t)-\Delta V(x,t) = f(x)e^{i\omega t}$, pour $t\to\infty$,

(b) des solutions de

(2) $\quad \Delta U_\varepsilon+(\omega^2+i\varepsilon)U_\varepsilon = f(x)$, pour $\varepsilon\to 0$.

On démontre simplement, à l'aide des formules de Kirchhoff, que

$$\lim_{t\to\infty} e^{-i\omega t}V(x,t) = \lim_{\varepsilon\to\infty} U_\varepsilon(x) = u(x),$$

$u(x)$ étant solution de $\Delta u(x)+\omega^2 u(x)=f(x)$.

Dans ce travail les auteurs donnent l'extension de ces résultats pour certaines classes générales d'équations elliptiques, à coefficients constants. Supposons $L(i^{-1}\partial/\partial x)$ $(x=(x_1,x_2,\cdots,x_n))$ tel que $L(s)\geq 0$ pour $s=(s_1,s_2,\cdots,s_n)$ réel, les coefficients de $L(s)$ ne dépendent pas de x, $L(s)$ est un polynome homogène de degré $2m$ dans les variables $(s_1,s_2,\cdots,s_n)$. On montre que si $V(x,t)$ est solution de $V_{tt}(x,t)+L(i^{-1}\partial/\partial x)V(x,t)=f(x)e^{i\omega t}$, telle que $V(x,0)=V_t(x,0)=0$, et si $n>2m$ (restriction essentielle), alors pour chaque x, on a que $\lim_{t\to\infty}V(x,t)e^{-i\omega t}=u(x)$, $u(x)$ étant solution de $Lu-\omega^2u=-f(x)$. Aussi, si $U_\varepsilon(x)$ est la solution unique dans $L^2(R^n)$ de l'équation $LU_\varepsilon-(\omega^2+i\varepsilon)U_\varepsilon=-f(x)$, on a pour chaque x que $\lim_{\varepsilon\to 0}U_\varepsilon(x)=u(x)$, $u(x)$ étant la même solution de l'équation $Lu-\omega^2u=-f(x)$.

Les démonstrations font usage extensive de la théorie des transformations de Fourier pour les distributions, telle qu'elle est exposée dans les deux premiers tomes de Gel'fand et Šilov sur les "fonctions généralisées" [Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; MR **20** #4182; **21** #5142a].

S. Zaidman (Bucharest)

6933:

Nieto, José. Eine Charakterisierung der elliptischen Differentialoperatoren. Math. Ann. **141**, 22–42 (1960).

Nous dirons qu'un opérateur différentiel $P(x,D)$ d'ordre m, défini dans un ouvert $\Omega\subset R^n$, est quasi-elliptique si $P\varphi\in\mathscr{L}_\Omega{}^0$ entraîne $\varphi\in\mathscr{L}_\Omega{}^m$. $\mathscr{L}_\Omega{}^0$ est l'espace des fonctions $\varphi(x)$ à carré sommable sur toute partie compacte de Ω. $\mathscr{L}_\Omega{}^m$ est l'espace des fonctions $\varphi(x)$ telles que $D^\nu\varphi(x)\in\mathscr{L}_\Omega{}^0$, pour $|\nu|\leq m$, muni des semi-normes $p_K(\varphi)$: $p_K{}^2(\varphi)=\sum_{|\nu|\leq m}\int_K|D^\nu\varphi(x)|^2dx$, K parcourant tous les compacts dans Ω. Il est un espace de Fréchet.

L'A. montre que P (supposé à coefficients indéfiniment différentiables) est quasi-elliptique si et seulement si P est elliptique. Voici sa démonstration en bref. Supposons P quasi-elliptique. On munit $\mathscr{L}_\Omega{}^m$ des semi-normes $q_K(\varphi)$: $q_K{}^2(\varphi)=\int_K(|\varphi|^2+|P\varphi|^2)dx$. Il montre que cette topologie est identique à celle de $\mathscr{L}_\Omega{}^m$ donnée auparavant, et ensuite que, si P n'est pas elliptique, cette situation n'arrive jamais. A la fin, il donne un exemple d'opérateur hypoelliptique, dont les opérateurs tangentiels sont nulle part hypo-elliptiques.

S. Mizohata (New York)

6934:

Boyarskiĭ, B. V. A general representation of the solutions of an elliptic system of $2n$ equations on a plane. Dokl. Akad. Nauk SSSR **122** (1958), 543–546. (Russian)

Si considerano in un dominio piano T di variabile complessa $z=x+iy$ i sistemi di equazioni del primo ordine di forma $(*)$ $u_x=\bar{A}u_y+\bar{B}u+f$, ove u è un vettore reale sconosciuto con $2n$ componenti, $\bar{A}$ e $\bar{B}$ sono matrici quadrate d'ordine $2n$ e f è un vettore reale, noti. L'ellitticità del sistema $(*)$ si esprime col fatto che in ogni punto del dominio T l'equazione del $(\bar{A}-\lambda E)=0$ non ne ha radici reali.

Sia poi (1) $w_{\bar{z}}-Qw_z=Aw+B\bar{w}+F$, la trascrizione complessa di forma canonica del sistema $(*)$, ove $w(z)$ è un vettore incognito complesso ad n componenti; A e B sono matrici quadrati complesse d'ordine n; F e un vettore complesso; Q è una matrice quasidiagonale del tipo $Q=\{Q_1,Q_2,\cdots,Q_p\}$; $Q_l=\{q_{lk,l}\}$—matrici quadrate, $q_{lk,l}=0$ per $k>l$, $q_{ll,l}=q_l$ e $q_{lk,l}=\beta_l{}^{l-k}$ per $k<l$; e q_l e β_l—funzioni complesse, definite nel T. L'A. elabora la teoria dei sistemi (1), parallela a quella dell'I. N. Vekua [Mat. Sb. **31** (**73**), (1952), no. 2, 217–314; MR **15**, 230], escogitata pel caso $n=1$ e $Q\equiv 0$, collegandola col seguente sistema di equazioni integrali del tipo di Fredholm

$$w(z)+\pi^{-1}\iint_K V(t,z)(Aw+B\bar{w})\,dK_t = \Phi(z),$$

ove $\Phi_{\bar{z}} - Q\Phi_z = 0$, e cioè Φ è un vettore Q-olomorfo (vedasi qui sotto). Il caso $n > 1$ differisce da quello in cui $n = 1$ in una serie di punti; tutti questi si collegano al fatto che per i sistemi (1) nel caso generale $n > 1$ il teorema di Liouville perde la sua generalità.

Risultati ottenuti, che ne trovano applicazione nella teoria dei problemi al contorno per sistemi (1), hanno alla base la teoria dei vettori Q-olomorfi, già stabilita nelle ipotesi alquanto più ristrette dall'A. Douglis [cf. Comm. Pure Appl. Math. **6** (1953), 259–289; MR **16**, 257], completamente analoga con quella dei vettori olomorfi.

D. Mangeron (Iaşi)

6935:

Boyarskiĭ, B. V. Some boundary value problems for systems of $2n$ elliptic equations in a plane. Dokl. Akad. Nauk SSSR **124** (1959), 15–18. (Russian)

This paper is one of a sequence devoted to the solution of an elliptic system of equations of the form $w_{\bar{z}} - Qw_z = Aw + B\bar{w} + F$. The unknown w is a vector of n complex valued functions. Q is a triangular matrix supposed to vanish outside some large circle and to admit generalized derivatives $Q_z, Q_{\bar{z}}$ in L_p, $p > 2$. The elements of the matrices A and B are functions in L_p. Let T be a multiply connected domain in the complex plane with $m+1$ Liapounof boundary curves. In T the above system is assumed strongly elliptic. In the problem of Riemann a solution is sought which is Hölder continuous in $T+L$ and satisfies $\operatorname{Re}[\overline{G(t)} \cdot w(t)] = f(t)$ on L. For the problem of Hilbert, solutions $w^+(t)$ are sought in $T+L$ and $w^-(t)$ in $E-T$ so that $w^+(t) = G(t)w^-(t) + h(t)$ on L. These problems are reduced to a discussion of certain singular integral equations. It is shown that a necessary and sufficient condition that the inhomogeneous Hilbert problem H admit a solution is that $\operatorname{Im} \int_L (h(t), (Edt + Q'\overline{dt})f) = 0$ for all solutions f of the adjoint homogeneous Hilbert problem H_0'. An analogous result holds for the Riemann problem. A relation is found between the numbers l_R and l_R' of solutions of the Riemann problem and its adjoint: $l_R - l_R' = 2\kappa - n(m-1)$, where $\kappa = (1/2\Pi)\Delta_L \arg\det G(t)$. [See preceding review and A. V. Bicadze, same Dokl. **112** (1957), 983–986; MR **20** #1079.]

A. N. Milgram (Berkeley, Calif.)

6936:

Godunov, S. K. Non-unique "blurrings" of discontinuities in solutions of quasilinear systems. Dokl. Akad. Nauk SSSR **136** (1961), 272–273 (Russian); translated as Soviet Math. Dokl. **2**, 43–44.

This paper is concerned with the theory of ordinary differential equations describing "blurrings" of discontinuities in quasilinear hyperbolic systems. Such blurrings are realized by replacing the zeros on the right-hand side of the system $\partial F_i(q_1, \cdots, q_n)/\partial t + \partial G_i(q_1, \cdots, q_n)/\partial t = 0$ by the viscous terms $\partial(\varepsilon \sum b_{ik}\, \partial q_k/\partial x)/\partial x$. Ordinary equations are obtained if we seek solutions in the form $q_j = q_j((t-\alpha x)/\varepsilon)$. It is shown that there can exist completely reasonable $\|b_{ik}\|$ for which the solution describing a blurred discontinuity (shock wave) will not be unique.

E. Leimanis (Vancouver, B.C.)

6937:

Gårding, Lars. Solution directe du problème de Cauchy pour les équations hyperboliques. La théorie des équations aux dérivées partielles. Nancy, 9–15 avril 1956, pp. 71–90. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXI. Centre National de la Recherche Scientifique, Paris, 1956. 187 pp. 1500 francs.

This paper describes the author's version of the method of energy inequalities for solving the Cauchy problem. As Friedrichs and Lewy did in their treatment of the second order case, Gårding starts with a quadratic Green's formula of the following kind: Let a, b denote two partial differential operators of order $m+1$ and m respectively, with variable coefficients. Let G be any domain, S its boundary. Then

$$\iint (bu)(au)\, dx = \int Au\, dS + \iint q(u)\, dx$$

where A is a quadratic form in the mth order partial derivatives of u, q in those of order $\leq m$. Suppose now that a is strictly hyperbolic; following Leray, choose b to be a hyperbolic operator of one lower order whose characteristics separate those of a, and choose G as a lens-shaped domain contained between two space-like surfaces. Then $\int A\, dS$ over each space-like surface is a positive definite quadratic functional; the proof of this fact employs a partition of unity and the use of Fourier transformation in the manner already employed by the author in his work on the positivity of quadratic functionals arising in the theory of elliptic operators. The second important ingredient of the proof is a representation of polynomials of one variable whose roots separate those of a given polynomial with real, distinct roots.

From such an energy identity one can easily deduce an energy inequality, which implies the uniqueness of solutions of the Cauchy problem. To construct solutions the author derives, with the aid of the Friedrichs mollifier technique, further inequalities about norms which are dual to the energy norm.

The paper is self-contained and the exposition lucid.

P. D. Lax (New York)

6938:

Gårding, Lars. Cauchy's problem for hyperbolic equations. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 104–109. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

This is a summary of the following two works by the author: (1) #6937 above. (2) *Cauchy's problem for hyperbolic equations*, corrected ed., Univ. of Chicago, Chicago, Ill., 1958 (mimeographed).

Let a denote a linear totally hyperbolic operator of order $n+1$, i.e., the generators of the characteristic cone are real and simple for all x, t. Let the coefficients of a be bounded, and let the coefficients of the principal part be Lipschitz continuous. Let the coefficient of $\partial^{n+1}/\partial t^{n+1}$ be 1. The method is based on the "generalized Friedrichs-Lewy inequality":

$$(1)\quad \left(\int \sum_{|\alpha| \leq m} |D^\alpha f(t,x)|^2 dx_1 \cdots dx_n\right)^{1/2} \leq c\left(\int \sum_{|\alpha| \leq m} |D^\alpha f(0,x)|^2 dx_1 \cdots dx_n\right)^{1/2} + c\int_0^t \left(\int |af(\tau,x)|^2 dx_1 \cdots dx_n\right)^{1/2} d\tau.$$

The theorems on the differentiability of the solutions are

in a sense optimal. The precise results are formulated as relations between function spaces.

A feature of the method is that approximation via Cauchy-Kowalevskaja is replaced by use of an inequality dual to (1), involving the adjoint a^*.

Partial adjoints are differential operators acting on pairs of functions. They enable the author to interpolate a sequence of Cauchy problems between a Cauchy problem for a and the corresponding Cauchy problem for a^*.

The type of equation here discussed does not seem to occur in applied mathematics, excepting the case $m=1$. However, the author states that the Cauchy problem for hyperbolic systems can be treated similarly. It would be desirable to present this variant of the theory also, especially because in the Chicago notes the elegance of the author's method is buried under a mountain of notation and generalization. *P. Ungar* (New York)

6939:

Dezin, A. A. Symmetric energy inequalities and a mixed problem. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 425–428. (Russian)

Es handelt sich um die Bestimmung allgemeiner Lösungen für das sogenannte "gemischte Problem" allgemeiner hyperbolischer Differentialgleichungen mit Hilfe gewisser Funktionale bestimmter Klasse. Im Falle daß die Koeffizienten (der Differentialgleichung) nicht von der Zeit abhängen, ist das analoge Problem von J. Lions behandelt worden [Acta Math. **94** (1955), 13–153; MR **17**, 745]. Verfasser benutzt eine von L. Gårding für das Cauchysche Problem entwickelte Methode [cf. #6937]. Nach Beweis einer Reihe von Hilfssätzen untersucht Verfasser das gemischte Problem für die Differentialgleichung $(*)$ $au = f$. Dabei bedeutet a den hyperbolischen Operator

$$a = \sum_{k,l=0}^{\nu}{}' D_k(a_{kl}D_l + a_k) + c, \quad D_k = \frac{\partial}{\partial x_k},$$

der über dem Bereich $[0 \leq x_0 \leq 1] \times \Omega$ definiert ist, wobei Ω einen endlichen Sternbereich im Raum der Veränderlichen $x_1, \cdots, x_n$ bezeichnet. Sodann wird für beliebige Elemente f eines Hilbertschen Raumes die Existenz und Eindeutigkeit einer Lösung der Differentialgleichung $(*)$ bewiesen und ihre Klassenzugehörigkeit bestimmt. Wenn in Einschränkung der für die Theorie zunächst bestehenden Voraussetzungen die Koeffizienten des Operators a von x_0 nicht abhängen, erfährt der Hilbertraum der Element f und deren Klassenzugehörigkeit eine entsprechende, von Verfasser präzisierte Modifikation.

M. Pinl (Cologne)

6940:

Kotake, Takeshi. Sur l'inégalité d'énergie pour l'équation différentielle p-parabolique. Proc. Japan Acad. **34** (1958), 681–686.

Le problème de Cauchy pour les équations p-paraboliques est déjà considéré par Eĭdel'man [Mat. Sb. (N.S.) **38** (**80**) (1956), 51–92; MR **17**, 857] et Mizohata [J. Math. Soc. Japan **8** (1956), 269–299; MR **19**, 285]. L'A. montre une autre voie, en démontrant deux inégalités d'énergie pour les opérateurs p-paraboliques. Ce sont des inégalités nouvelles dont la démonstration est de caractère élémentaire. Sa méthode est celle de Gårding utilisée pour les équations hyperboliques [voir #6937].

S. Mizohata (New York)

6941:

Kotake, Takeshi. Isomorphisme et le problème de Cauchy pour un opérateur différential p-parabolique. Mem. Fac. Engrg. Kyoto Univ. **21** (1959), 331–347.

Le problème de Cauchy pour les équations p-paraboliques à coefficients variables a été traité par S. D. Eĭdel'man [Math. Sb. N.S. **38** (**80**) (1956), 51–92; MR **17**, 857] par la construction explicite des noyaux élémentaires, et aussi par S. Mizohata [J. Math. Soc. Japan **8** (1956), 269–299; MR **19**, 285] par la méthode de Leray. Ces deux traitements ne sont pas trop simples. L'A. présente ici sa méthode en suivant la méthode de Gårding dévelopée pour les équations hyperboliques [voir #6937]. À partir de l'opérateur à coefficients constants, l'A. montre l'inégalité d'énergie d'une manière assez claire. Cette déduction, qui est peut-être inspirée par des travaux récents sur l'unicité du problème de Cauchy, est un point de clé de cet article. À la fin, la régularité au bord et l'hypo-ellipticité sont discutées. *S. Mizohata* (New York)

6942:

Milicer Grużewska, Halina. Le théorème d'unicité de la solution d'un système parabolique d'équations linéaires avec les coefficients hölderiens. Ricerche Mat. **9** (1960), 20–42.

A summary of this paper has appeared in Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 593–599 [MR **22** #3886]. Let (1) $L(x, t, \partial/\partial x)u - \partial u/\partial t = 0$ be a parabolic system of order M in the sense of Petrowski, with uniformly bounded and uniformly Hölder continuous coefficients, where x varies in the n-dimensional Euclidean space E^n and $0 \leq t \leq T$. Consider the class of solutions defined by (2): for $0 \leq k \leq M$, $\int_0^T \int_{E^n} |\partial^k u/\partial x_1^{k_1} \cdots \partial x_n^{k_n}|\, dx dt < \infty$. The author proves that if u is a solution of (1) which satisfies (2), and if $u(x, 0) \equiv 0$, then $u(x, t) \equiv 0$. The method is based on results of W. Pogorzelski [Ricerche Mat. **7** (1958), 153–185; Math. Scand. **6** (1958), 237–262; MR **21** #4300, #4302] concerning fundamental solutions. {A stronger theorem has been independently proved by Aronson [Bull. Amer. Math. Soc. **65** (1959), 310–318; MR **22** #3888], namely, the above uniqueness theorem holds with (2) replaced by the weaker class (2′): for $0 \leq k \leq M$, $\int_0^T \int_{E^n} |\partial^k u/\partial x_1^{k_1} \cdots \partial x_n^{k_n}| \exp(-\beta|x|^q) dx dt < \infty$, where $q = M/(M-1)$ and $\beta > 0$. It may be mentioned that for smooth coefficients whose derivatives, to a certain order, are uniformly bounded, the above uniqueness theorem holds with (2′) replaced by (2″): the inequality in (2′) corresponding to $k=0$ is satisfied. This was proved by Slobodeckiĭ [Math. Sb. (N.S.) **46** (**88**) (1958), 229–258; MR **22** #830] and the reviewer [Amer. J. Math. **81** (1959), 503–511; MR **21** #3657].}

A. Friedman (Minneapolis, Minn.)

6943:

Cattabriga, Lamberto. Una generalizzazione del problema fondamentale di valori al contorno per equazioni paraboliche lineari. Ann. Mat. Pura Appl. (4) **46** (1958), 215–247.

Consider a linear parabolic equation

$$(*) \qquad L(u) = \frac{\partial^4 u}{\partial x^4} + \frac{\partial u}{\partial y} + \sum_{i=0}^{3} a_i(x, y) \frac{\partial^i u}{\partial x^i} = F(x, y) \cdot$$

over a domain D of the form $X_1(y) \leq x \leq X_2(y)$, $0 \leq y \leq 1$. Let C_0, C_1, C_2 denote the lower horizontal, the left, and the right boundary curves of D, respectively. A solution of (*) is sought which has given values on C_0 and which, with its derivative $\partial u/\partial x$, also is given on C_1 and C_2. This problem has been previously solved by B. Pini [same Ann. **43** (1957), 261–297; MR **19**, 965], but under restrictive conditions which it is the author's intention to weaken. In this paper, the differential equation (*) is required to hold merely almost everywhere, and the boundary conditions also are appropriately generalized. If $F(x, y)$ is square integrable in D, the generalized problem is shown to have a solution obtainable, through the application of suitable a priori integral inequalities, as the limit of solutions of approximating regular problems. This solution is unique, as is shown with the aid of Green's function for the homogeneous equation. An example is given of a continuous F such that, although there are functions u satisfying (*) almost everywhere, no u exists for which (*) holds at every point of any open subset of D.

A. Douglis (College Park, Md.)

6944:

Cattabriga, Lamberto. Problemi al contorno per equazioni paraboliche di ordine $2n$. Rend. Sem. Mat. Univ. Padova **28** (1958), 376–401.

Consider a linear parabolic equation

$$(*) \qquad L_0(u) = \frac{\partial^{2n}u}{\partial x^{2n}} + (-1)^n \frac{\partial u}{\partial y} = 0$$

over a domain D of the form $X_1(y) \leq x \leq X_2(y)$, $0 \leq y \leq 1$. Let C_0, C_1, C_2 denote the lower horizontal, the left, and the right boundary curves of D, respectively. A solution of (*) is sought which has given values on C_0 and which is such that the derivatives $\partial^i u/\partial x^i$, $i = 0, 1, \cdots, n-1$, also are given on C_1 and C_2. This solution is obtained in a traditional manner with the aid of line potentials, involving the fundamental solution of the equation, analogous to the simple and double layers of the theory of heat flow. The properties of these potentials being thoroughly investigated, they are then applied to solving the same problem for the more general equation $L_0(u) + \sum_{i=0}^{2n-1} [a_i(x, y) \times(\partial^i u/\partial x^i)] = f(x, y)$ by a scheme used by B. Pini [Ann. Mat. Pura Appl. (4) **43** (1957), 261–297; MR **19**, 965] in the case $n = 2$. It is noted that, when f is square integrable, and the assumption of boundary data is permitted in a generalized sense, the author's method for the case $n = 2$ [see preceding review] applies here and yields solutions of this problem in a suitable generalized sense. Related results are stated concerning non-linear equations of parabolic type.

A. Douglis (College Park, Md.)

6945:

Èĭdel'man, S. D.; Porper, F. O. On certain properties of systems parabolic in the sense of G. E. Šilov. Dokl. Akad. Nauk SSSR **126** (1959), 948–950. (Russian)

Les auteurs obtiennent des théorèmes "de Liouville" et des théorèmes de stabilisation en donnant des estimations pour des matrices de Green correspondant aux classes assez générales des systèmes qui sont des sous-classes des systèmes paraboliques dans le sens de Šilov.

S. Mandelbrojt (Paris)

6946:

Delsarte, Jean. Hypergroupes et opérateurs de permutation et de transmutation. La théorie des équations aux dérivées partielles. Nancy, 9–15 avril 1956, pp. 29–45. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXI. Centre National de la Recherche Scientifique, Paris, 1956. 187 pp. 1500 francs.

Let V be a space of functions, or of distributions, defined on R^n or on C^n, and X, A, B be operators defined on V and with values also in V, with X being linear and continuous and A and B being differential operators. Definitions: X is said to be a "permutation operator with respect to A and B" provided that $AX = XB$. If, further, X is an isomorphism of V, then X is called a "transmutation operator". Example: Suppose that $F(x, y)$ is a nonzero solution of the partial differential equation "with separated variables" ${}^tA_x F(x, y) = B_y F(x, y)$, where ${}^tA$ denotes the "transpose", or adjoint, operator of A, and the notation $B_y F(x, y)$ means that the operator B "acts" on the variable y, with x being regarded merely as a parameter. Then the operator X which takes $\varphi \in \mathscr{D}(R_x^n)$ into $\int F(x, y)\varphi(x)dx$ is a permutation operator with respect to A and B. The purpose of this paper is to give a general process for the construction of permutation operators, with specific illustrations of the general procedure. This is carried out using the notion of a "hypergroup"; which, roughly speaking, consists of two algebras of operators S_a, T_b with subscripts a, b belonging to a Lie group G, which act on functions defined on G, and are such that (1) S_a and T_b commute for any a, b in G; (2) one has $S_e = T_e = E$, the identity operator, e being the neutral element of G; and (3) one has $(S_a f)_x = (T_x f)_a$, where $\mathfrak{G}$ is a compact group of automorphisms of G, and f is a continuous function defined on the quotient group $G/\mathfrak{G}$.

J. B. Diaz (College Park, Md.)

6947:

Ladyženskaya, O. A. (Editor). Mathematical questions of hydrodynamics and magnetohydrodynamics for viscous incompressible fluids. Trudy Mat. Inst. Steklov. **59** (1960). 189 pp. (Russian)

This volume is composed of seven papers on the subject, which will be given independent reviews in MR.

6948:

Eremin, S. A. Solutions of a type of differential equations related to entire functions of two complex variables. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 316–323. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The author extends Gel'fond's work on differential equations of infinite order with constant coefficients [Trudy Mat. Inst. Steklov. **38** (1951), 42–67; MR **13**, 929] to partial differential equations with constant coefficients,

$$\sum \sum a_{mn} \frac{\partial^{m+n} F(w, z)}{\partial w^m \partial z^n} = \Phi(w, z).$$

When the characteristic function $\sum \sum a_{mn} \xi^m \zeta^n$ is regular at the origin and Φ is of sufficiently slow growth there is a solution that can be represented by a series of polynomials.

R. P. Boas, Jr. (Evanston, Ill.)

POTENTIAL THEORY

See also B7641.

6949:

Górski, J. Une remarque sur la méthode des points extrémaux de F. Leja. Ann. Polon. Math. 7 (1959), 63–69.

Dans R^2 soit E compact de capacité >0 et soit $f(z)$ réelle finie continue sur E. On pose avec Leja $\omega_\lambda(z, \zeta)=|z-\zeta| \times \exp(-\lambda(f(z)+f(\zeta)))$ (λ paramètre); $\prod_{0\le j<k\le n} \omega_\lambda(\zeta_j, \zeta_k)$, pour les systèmes de $n+1$ points $\zeta_j \in E$, atteint son maximum pour des systèmes de points dits extrémaux. Une certaine fonction de Leja formée à partir de ces points admet, pour $n \to \infty$, une limite $\Phi(z, \lambda f)$. On sait [voir Leja, Ann. Soc. Polon. Math. **23** (1950), 204–205; MR **12**, 703; Inoue, Proc. Imp. Acad. Tokyo **14** (1936), 368–372; Górski, mêmes Ann. **1** (1955), 418–429; MR **17**, 604] que $\lambda^{-1} \log \Phi(z, \lambda f)$ tend ($\lambda \to 0$) vers $\psi(z)$, solution du problème de Dirichlet pour le domaine D limité par E, lorsque E est la frontière commune de D borné et d'un domaine contenant le point à l'infini.

L'A. avait déjà montré pour certains f très particuliers, que le passage à la limite ($\lambda \to 0$) est inutile. Il montre ici que pour E et f assez réguliers, $\lambda^{-1} \log \Phi$ représente la solution quel que soit $\lambda > 0$ assez petit. Considérations du même ordre dans R^3. *M. Brelot* (Urbana, Ill.)

6950:

Zeuli, Tino. Sui potenziali poliarmonici in un campo sferico oppure circolare. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **93** (1958/59), 313–339.

L'auteur calcule explicitement et sans quadratures les potentiels premier et second $\int ur^{2s-1}d\sigma$ ($n=0, 1$), d'une couche sphérique dont la densité superficielle u est un polynôme, ou d'une distribution en volume à l'intérieur d'une sphère avec densité u polyharmonique. Des calculs analogues sont faits pour le cercle et le potentiel logarithmique, ordinaire $\int u \log r^{-1}d\sigma$, ou second $\int u(1+\log r^{-1})r^2 d\sigma$. *L. Naïm* (Grenoble)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 7077.

6951:

Naftalevič, A. G. A system of two difference equations. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 217–225. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

Given the system

$$f(z+n\alpha) = \sum_{k=0}^{n-1} p_k(z)f(z+k\alpha)+a(z),$$

$$(1) \qquad f(z+m\beta) = \sum_{l=0}^{m-1} q_l(z)f(z+l\beta)+b(z),$$

$$p_0(z) \neq 0, \quad q_0(z) \neq 0, \quad \operatorname{Im} \alpha/\beta \neq 0,$$

where z runs through all integer points of the complex plane; values of the function $f(z)$ (initial data) are given on integer points of a closed rectangle with the vertices z, $z+(n-1)\alpha$, $z+(n-1)\alpha+(m-1)\beta$, $z+(m-1)\beta$ (z an integer point). It is required to find the function $f(z)$ as the solution of (1) on the whole integer lattice of the complex plane.

The author presents necessary and sufficient conditions for (1) to be compatible, i.e., for a solution to exist when the initial values and the initial rectangle have been selected. In two cases is proved the existence of a solution with an analytic character: when $m=n=1$ and when the system has constant coefficients. It is proved that the homogeneous system (1) with constant coefficients (i.e., with $a(z)=b(z)=0$) generally has no entire solutions except the trivial solution.

Sets of asymptotic periods are investigated, and results due to Whittaker [*Interpolatory function theory*, Cambridge Univ. Press, London, 1935] and to Gel'fond [Isčislenie konečynh raznosteĭ, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952; MR **14**, 759] are proved. *M. D. Friedman* (Needham Heights, Mass.)

6952:

Kordylewski, J.; Kuczma, M. On the functional equation $F(x, \varphi(x), \varphi[f_1(x)], \cdots, \varphi[f_n(x)])=0$. Ann. Polon. Math. 8 (1960), 55–60.

Die Verff. beweisen, dass die im Titel figurierende Funktionalgleichung unendlich viele stetige Lösungen hat, falls die Funktionen $f_i(x)$ ($i=1, 2, \cdots, n$) stetig und streng wachsend in $[a, b]$ sind, $f_i(a)=a$, $f_i(b)=b$ ($i=1, 2, \cdots, n$), $x<f_1(x)\le f_i(x)\le f_{n-1}(x)<f_n(x)$ ($x \in (a, b)$, $i=2, \cdots, n-2$), falls $F(x, y_0, y_1, \cdots, y_n)$ stetig in einer Menge Ω des $(n+2)$-dimensionalen Raumes ist und die nichtleere Menge der Punkte $(x, y_0, y_1, \cdots, y_n) \in \Omega$ mit $F(x, y_0, y_1, \cdots, y_n)=0$ die Würfel $a<x<b$, $c<y_i<d$ ($i=0, \cdots, n-1$) bzw. $a<x<b$, $c<y_j<d$ ($j=1, \cdots, n$) als Projektionen hat, und endlich $F(x, y_0, y_1, \cdots, y_n)=0$ in Ω bezüglich y_0 und y_n lösbar ist.—Der Beweis ist einfach und ähnelt dem in einem Spezialfall von den Verff. in Ann. Polon. Math. 7 (1959), 21–32 [MR **22** #850] gegebenen. *J. Aczél* (Debrecen)

SEQUENCES, SERIES, SUMMABILITY

See also 6783, 6887.

6953:

Wynn, P. Converging factors for continued fractions. I, II. Numer. Math. 1 (1959), 272–320.

The continued fractions considered are of the form

$$u_0 = \frac{a_0}{b_0+}\frac{c_0}{d_0+}\cdots\frac{y_0}{+z_0+}\frac{a_1}{b_1+}\frac{c_1}{d_1+}\cdots\frac{y_1}{+z_1+}\frac{a_2}{b_2+}\cdots,$$

where the elements $a_n, b_n, \cdots, z_n$, are polynomials in n. For such continued fractions the tail

$$u_n = \frac{a_n}{b_n+}\frac{c_n}{d_n+}\cdots\frac{y_n}{+z_n+}\frac{a_{n+1}}{b_{n+1}+}\cdots,$$

satisfies a Riccati difference equation

$$p(n)u_n+q(n)u_{n+1}+r(n)u_nu_{n+1} = s(n),$$

where p, q, r, s, are polynomials in n. In general there are two solutions $u_n^{(1)}$ and $u_n^{(2)}$ of the form $u_n=\sum_{s=-k}^{\infty} \alpha_s n^{-s}$

for the Riccati difference equation; these are called converging factors for the continued fraction u_0. Since $u_0 = T_n(u_n)$ where T_n is a linear fractional function, the value of u_0, when convergent, is approximated by the subsequence $\{T_n(0)\}$ of approximants. The major point of the paper is that if v is a section of the series for a suitably chosen converging factor $u_n^{(1)}$, then $T_n(v)$ is, generally speaking, a much better approximation to u_0 than is $T_n(0)$. This fact is illustrated by a number of examples.

W. T. Scott (Evanston, Ill.)

6954:

Ul'yanov, P. L. Strongly unconditionally convergent series. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 75–92. (Russian)

The paper contains many results about convergence and divergence of series of functions, of which we give some examples. A series (*) $\sum_{n=1}^{\infty} f_n(x)$, $x \in E$, is called strongly unconditionally convergent (s.u.c.) on E if each rearrangement of the series converges everywhere on E except for a countable subset. Generalizing a result of Besicovitch [J. London Math. Soc. **28** (1953), 480–483; MR **15**, 117] the author shows that if the continuum hypothesis is true and if E has the power of continuum, then there exists a series (*) which together with all its infinite subseries is s.u.c. on E. If the functions $f_n(x)$ are measurable on a set E of positive measure of the real line and (*) is s.u.c. on E, then $\sum |f_n(x)| < +\infty$ almost everywhere on E. Assume that $\varphi(x)$ has period 1 and is not identically zero. If each rearrangement of $\sum a_n \varphi(\lambda_n x + \alpha_n)$ converges on a subset of second category of $[0, 1]$, then $\sum |a_n| < +\infty$. There are also some theorems about lacunary trigonometric series.

G. G. Lorentz (Syracuse, N.Y.)

6955:

Šalát, Tibor. Absolut konvergente Reihen und das Hausdorffsche Mass. Czechoslovak Math. J. **9** (84) (1959), 372–389. (Russian summary)

Sei $\sum_{n=1}^{\infty} a_n$ eine konvergente Reihe mit positiven Gliedern, bei der jedes Restglied $R_k = \sum_{n=k+1}^{\infty} a_n$ $(k=1, 2, \cdots)$ der Bedingung $R_k < a_k$ genügt. Verf. untersucht die Menge W aller Zahlen der Form $x = \sum_{n=1}^{\infty} b_n$, $b_n = \pm a_n$, und beweist folgende Abschätzung für deren Hausdorffsche Dimension: (1) Es gilt $\log 2/\log S_1 \leq \dim W \leq \log 2/\log S_2$, wobei

$$S_1 = 1 + \limsup_{k\to\infty} (a_k/R_k), \quad S_2 = 1 + \liminf_{k\to\infty} (a_k/R_k)$$

ist.

In dem Fall, dass das Lebesguesche Mass $\mu(W)$ positiv ist, wird die Vorzeichenverteilung der Summanden b_n näher untersucht, und es gelten analoge Sätze wie bei der Ziffernverteilung dyadischer Brüche. Verf. beweist in diesem Fall: (2) Sei $0 < \zeta < \frac{1}{2}$ und $W(\zeta)$ die Menge der $x \in W$, bei denen die Anzahl $f(n, x)$ der positiven b_k mit $k \leq n$ der Ungleichung $\liminf_{n\to\infty} f(n, x)/n \leq \zeta$ genügt. Dann ist $\dim W(\zeta) = (\zeta \log \zeta + (1-\zeta)\log(1-\zeta))/\log \frac{1}{2}$. Für den Fall $\mu(W) = 0$, wo die analoge Frage offen bleibt, wird ersatzweise eine untere Schranke für $\dim \overline{W}$ angegeben, wenn $\overline{W}$ die Menge aller $x \in W$ mit $\lim_{n\to\infty} f(n, x)/n = \frac{1}{2}$ bezeichnet.

Die Beweise von (1) und (2) lehnen sich methodisch an eine Arbeit des Ref. [J. Reine Angew. Math. **190** (1952), 199–230; MR **15**, 15] an, die in dem für (2) relevanten Teil eine Fortsetzung älterer Untersuchungen von Knichal und Besicovitch darstellt.

B. Volkmann (Mainz)

6956:

Vernotte, Pierre. A propos de la sommation pratique des séries divergentes. C. R. Acad. Sci. Paris **250** (1960), 1431–1432.

La série $\sum u_n$ est sommée en calculant par des procédés empiriques le prolongement analytique de $f(z) = \sum u_n^{-z}$ supposée holomorphe dans Re $z > 0$. Suivant l'Auteur "la justification complète est encore à apporter". On ne peut agréer avec l'Auteur que sur ce point.

P. Malliavin (Princeton, N.J.)

6957:

Borwein, D. On methods of summability based on integral functions. II. Proc. Cambridge Philos. Soc. **56** (1960), 125–131.

Diese Arbeit ist eine Fortführung von zwei früheren Arbeiten des Verf. [Proc. Roy. Soc. Edinburgh. Sect. A **64** (1957), 342–349; Proc. Cambridge Philos. Soc. **55** (1959), 23–30; MR **19**, 955; **21** #245]. Der Verf. behandelt hier Vergleichssätze der Form $Q^* \subseteq P^*$ und $Q \dot{\subseteq} P^*$ (dies bedeutet, dass $s_n \to l\ (P)^*$ gilt, wenn $s_n \to l\ (Q)$ und der Konvergenzradius von $\sum p_n s_n z^n$ grösser als Null ist). U.a. wird gezeigt, dass für das durch $\sum_{n=N}^{\infty} z^n/\Gamma(\alpha n+\beta)$ ($\alpha > 0$, β reell) gegebene Verfahren (B, α, β) gilt $(B, a, b)^* \dot{\supseteq} (B, \alpha, \beta)$ ($\alpha > a > 0$, β, b reell), $(B, \alpha, b) \supseteq (B, \alpha, \beta)$, und $(B, \alpha, b)^* \supseteq (B, \alpha, \beta)^*$ ($\alpha > 0$, $b > \beta$). Die zu

$$\sum_{n=N}^{\infty} \frac{z^n}{\{\Gamma(an+b)\}^c (n+p)^{qn+r}}$$

($a > 0$, $ac+q > 0$) gehörigen Verfahren P und P^* sind äquivalent zu $(B, ac+q, bc+r-\frac{1}{2}c+\frac{1}{2})$ und $(B, ac+q, bc+r-\frac{1}{2}c+\frac{1}{2})^*$.

A. Peyerimhoff (Marburg)

6958:

Borwein, D. Relations between Borel-type methods of summability. J. London Math. Soc. **35** (1960), 65–70.

Es werden die Untersuchungen einer früheren Arbeit [Mathematika **5** (1958), 128–133; MR **22** #2815; vgl. auch das vorangehende Referat] fortgeführt. Ist

$$s_n = \sum_{r=0}^{n} a_r, \quad a(x) = \sum_{n=N}^{\infty} \frac{a_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}, \quad s(x) = \sum_{n=N}^{\infty} \frac{s_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}$$

(mit Konvergenzradius von $\sum a_n z^n/\Gamma(\alpha n+\beta)$ und $\sum s_n z^n/\Gamma(\alpha n+\beta)$ grösser als Null) und sind $s^*(x)$, $a^*(x)$ analytische Fortsetzung von $s(x)$, $a(x)$ für alle $x > 0$, so ist $\sum a_n$ und $\{s_n\}$ zum Wert l $(B, \alpha, \beta)^*$ bzw. $(B', \alpha, \beta)^*$ summierbar, wenn gilt $\alpha e^{-x} s^*(x) \to l$ bzw. $\int_0^x e^{-t} a^*(t)\,dt + s_{N-1} \to l$ $(x \to \infty)$. Verf. zeigt, dass (1) $\sum a_n = l\ (B, \alpha, \beta)^*$ genau dann, wenn $\sum a_n = l\ (B', \alpha, \beta)^*$ und $a_n \to 0\ (B, \alpha, \beta)^*$, (2) $(B, \alpha, \beta+1)^* \cong (B', \alpha, \beta)^*$.

A. Peyerimhoff (Marburg)

6959:

Borwein, D. On moment constant methods of summability. J. London Math. Soc. **35** (1960), 71–77.

Sei $\chi(x) \nearrow$ und beschränkt für $0 \leq x < X \leq \infty$, $\chi(y) < \lim_{x\to X-} \chi(x)$ $(0 \leq y < X)$ und $0 < \mu_n = \lim_{x\to X-} \int_0^x t^n d\chi(t)$ $(n = 0, 1, \cdots)$, $\sum_{n=0}^{\infty} x^n/\mu_n = M(x)$ mit Konvergenzradius R. Ist $s_n = \sum_{r=0}^{n} a_r$, $S(z) = \sum_{n=0}^{\infty} s_n z^n/\mu_n$, $A(z) = \sum_{n=0}^{\infty} a_n z^n/\mu_n$ (Konvergenzradius > 0) und sind $S^*(x)$, $A^*(x)$ analytische Fortsetzungen für $(0, R)$, so gilt $\sum_0^{\infty} a_n = l\ (J_\chi)^*$ und

$s_n \to s\ (J_\chi)^*$, wenn $S^*(x)/M(x) \to l$ für $x \to R-$, ferner $\sum_0^\infty a_n = l\ (J_\chi')^*$ und $s_n \to s\ (J_\chi')^*$, wenn

$$\lim_{x \to X-} \int_0^x A^*(t)\, d\chi(t) = l.$$

Der Verf. zeigt, dass für das durch $\chi(x) = -(1-x)^\alpha$ $(\alpha > 0,\ 0 \leqq x < X = 1)$, $R = 1$ erklärte Verfahren A_α gilt: (1) $\sum_0^\infty a_n = l\ (A_\alpha)^*$ gilt genau dann dann, wenn $\sum a_n = l$ $(A_\alpha')^*$ und $na_n \to 0\ (A_{\alpha-1})^*$ (mit einer weiteren Definition von A_α für $\alpha < 0$). (2) $(A')^* \cong (A_{\alpha-1})^*$. Ein entsprechender Satz für Borelverfahren wurde vom Verf. in der im vorangehenden Referat besprochenen Arbeit bewiesen. *A. Peyerimhoff* (Marburg)

6960:

Andersen, A. F. A condition for C-summability of negative order. Math. Scand. **7** (1959), 337–346.

Der Verf. zeigt, dass das Cauchy-sche Konvergenzkriterium auf $C_{-\delta}$-Summierbarkeit $(0 \leqq \delta < 1)$ ausgedehnt werden kann: $\sum u_n$ ist genau dann $C_{-\delta}$-summierbar, wenn zu jedem $\varepsilon > 0$ ein $N(\varepsilon)$ existiert, so dass gilt

$$\left| \sum_{\mu=\nu}^{n} u_\mu \binom{n-\mu-\delta}{n-\mu} \right| < \varepsilon n^{-\delta}$$

für alle $n \geqq \nu > N(\varepsilon)$. Als Anwendung wird gezeigt: Ist $\sum \binom{\mu+r-1}{\mu} a_\mu$ $C_{-\delta}$-summierbar für $0 < \delta < 1$, wobei $0 < r < 1$, so ist $\Delta^{-r} a_\nu = o(\nu^{1-r-\delta})$ für $0 < \delta < 1-r$, $\Delta^{-r} a_\nu = o(\log \nu)$ für $\delta = 1-r$, $\Delta^{-r} a_\nu = o(1)$ für $1-r < \delta < 1$.
A. Peyerimhoff (Marburg)

6961:

Kangro, G.; Baron, S. Summierbarkeitsfaktoren für Cesàro-summierbare und Cesàro-beschränkte Doppelreihen. Tartu Riikl. Ül. Toimetised **73** (1959), 3–49. (Russian. Estonian and German summaries)

Zahlen ε_{mn} heissen vom Typ (A, B), wenn jede A-summierbare Reihe $\sum u_{mn}$ in eine B-summierbare Reihe $\sum u_{mn}\varepsilon_{mn}$ übergeführt wird. Die Verff. betrachten die Verfahren $C^{\alpha\beta}, C_b^{\alpha\beta}, C_r^{\alpha\beta}, C_0^{\alpha\beta}$ (d.h. die $C^{\alpha\beta}$-Transformation sei konvergent, konvergent und beschränkt, regulär konvergent, beschränkt) und geben für $\alpha \geqq 0$, $\beta \geqq 0$ ganz, $\gamma \geqq 0$, $\delta \geqq 0$ reell, genaue Bedingungen für Zahlen ε_{mn} der Typen $(C_r^{\alpha\beta}, B)$, $(C_b^{\alpha\beta}, B)$, $(C_0^{\alpha\beta}, B)$—wo B eines der Verfahren $C^{\gamma\delta}$, $C_b^{\gamma\delta}$, $C_r^{\gamma\delta}$ sei—an. Für $(C_r^{\alpha\beta}, C^{\gamma\delta})$ $0 \leqq \gamma$, $\delta \leqq \alpha, \beta$ ist z.B. notwendig und hinreichend, dass gilt $\sum_{m,n} (m+1)^\alpha (n+1)^\beta |\Delta_{mn}{}^{\alpha+1,\beta+1}\varepsilon_{mn}| < \infty$, $\sum_m (m+1)^\alpha \times |\Delta_m{}^{\alpha+1}\varepsilon_{mn}| = O(1)(n+1)^{\delta-\beta}$, $\sum_n (n+1)^\beta |\Delta_n{}^\beta \varepsilon_{mn}| = O(1) \times (m+1)^{\gamma-\alpha}$; $\varepsilon_{mn} = O(1)(m+1)^{\gamma-\alpha}(n+1)^{\delta-\beta}$. Schliesslich werden verschiedene spezielle Folgen untersucht.
A. Peyerimhoff (Marburg)

6962:

Rhoades, B. E. Total comparison among some totally regular Hausdorff methods. Math. Z. **72** (1959/60), 463–466.

Es sei $\Gamma_a{}^k$ das durch die Diagonalfolge $[a/(n+a)]^k$ erzeugte Hausdorff-Verfahren $(a, k > 0)$. Nachdem Basu in verschiedenen Arbeiten [Proc. London Math. Soc. (2) **50** (1949), 447–462; Math. Z. **67** (1957), 303–309; MR **10**, 368; **19**, 955] die Limitierungsverfahren H_k und C_k, sowie H_k und $\Gamma_a{}^k$ total verglichen hatte, das heißt mit Einschluß der Limitierbarkeit nach $+\infty$, gibt der Verf. hier die entsprechenden Verhältnisse zwischen C_k und $\Gamma_a{}^k$ an. Beispiel: Für $0 < 1+k \leqq a < 1$ ist $\Gamma_a{}^k$ total stärker als C_k, aber nicht umgekehrt. *D. Gaier* (Pasadena, Calif.)

6963:

Kuttner, B. Some theorems on Mercer's and other related transformations. Quart. J. Math. Oxford Ser. (2) **11** (1960), 151–160.

The relation between Hausdorff methods (H, μ_n) and quasi-Hausdorff methods (H^*, μ_{n+1}) has been studied by Ramanujan [same J. (2) **8** (1957), 197–213; MR **20** #7167] and Kuttner [ibid., 272–278; MR **20** #7168]. For any complex number λ, let $H(\lambda) = \lambda I + (1-\lambda)C$ where I, C denote, respectively, the identity and the Cesàro transformation $(C, 1)$. Then C is a Hausdorff method (H, ν_n) with $\nu_n = 1/(n+1)$ and is equivalent to $C^* \equiv (H^*, \nu_{n+1})$. Kuttner [J. London Math. Soc. **34** (1959), 401–405; MR **21** #5834] proved (I) $H(\lambda)$ is equivalent to convergence if and only if $\operatorname{re} \lambda > 0$, whereas the corresponding quasi-Hausdorff matrix $H^*(\lambda) = \lambda I + (1-\lambda)C^*$ is equivalent to convergence if and only if either $\operatorname{re} \lambda < 0$, or $\lambda \neq 0$, $\operatorname{re}(1/\lambda) \geqq 1$. In the present paper, the author proves (II) if $\{s_n\}$ is Abel-summable and is also summable $H(\lambda)$ [summable $H^*(\lambda)$], $\lambda \neq 0$, then $\{s_n\}$ is convergent. {Jakimovski and Parameswaran [Quart. J. Math. Oxford Ser. (2) **9** (1958), 290–298; MR **21** #4318] proved this result for $H(\lambda)$, $\lambda \neq 0$, $\operatorname{re} \lambda \neq 0$, and the latter [ibid. **10** (1959), 224–229; MR **22** #155] gives the analogue of (II) for summability by the Borel and $H(\lambda)$ methods.} The author deduces part of (II) from the more general theorem: (III) Let $\{s_n\}$ be summable $H(\lambda)$ [summable $H^*(\lambda)$] to 0, where $\operatorname{re} \lambda = 0$, $\lambda \neq 0$. Then $\psi(x) = (1-x) \sum s_n x^n$ converges for $0 < x < 1$ and $\psi(1-1/n) - R_n \sigma_n \to 0$ as $n \to \infty$, where $\{\sigma_n\}$ is the C-transform [C^*-transform] of $\{s_n\}$ and $\lim_{n\to\infty} R_n$ exists.

Function analogues of (I), (II) and (III) are also proved. *M. R. Parameswaran* (Madras)

6964:

Jeśmanowicz, L. On the Hardy-Landau theorem. Colloq. Math. **7** (1959/60), 261–264.

Der Verfasser gibt einen weiteren Beweis des $O_L - C_k \to K$-Satzes, wobei allerdings die Äquivalenz $C_k \approx HC_{k-1}$ verwendet wird; siehe auch Gaier und Zeller [Rend. Circ. Mat. Palermo (2) **3** (1954), 83–88; MR **16**, 124.]
D. Gaier (Pasadena, Calif.)

6965:

Davydov, N. A. More on the converse of Abel's theorem. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 29–34. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The author announces generalisations of his earlier Tauberian theorems for Abel's method [Mat. Sb. (N.S.) **39** (**81**) (1956), 401–404; MR **18**, 733]. The main improvement is the replacement of the condition "$f(z)$ is continuous in $|z-a| \leqq 1-a$ $(0 \leqq a < 1)$" by the condition "$f(z)$ bounded in $|z-a| < 1-a$, $f(x) \to f(1)$ as $x \to 1-0$ along the positive real axis". *W. H. J. Fuchs* (Ithaca, N.Y.)

6966:

Kull', I. G. Multiplication of summable double series. Tartu Riikl. Ül. Toimetised **62** (1958), 3–59. (Russian. Estonian and English summaries)

Der Verf. betrachtet eine Reihe von Räumen aus Doppelfolgen $x=\{\xi_{mn}\}$: c ($\lim_{mn} \xi_{mn}$ existiert); r ($x\in c$ und $\lim_m \xi_{mn}$ und $\lim_n \xi_{mn}$ existieren); c_λ ($\lim \xi_{mn}$ existiert für $1/\lambda \leqq m/n \leqq \lambda$); m, $\tilde{m}$, m_λ ($\xi_{mn}=O(1)$ für alle m, n; für $m, n \geqq N(x)$; für $1/\lambda\leqq m/n\leqq\lambda$); l ($\sum|\xi_{mn}|<\infty$); $c_{\varphi\psi}{}^z$, $c_{\varphi\psi}{}^c$, $c_{\varphi\psi}{}^m$ ($x\in c$ und $\xi_{mn}/\varphi(m)$, $\xi_{mn}/\psi(n)$ konvergiert gegen Null; konvergiert; ist beschränkt; $\varphi(m)$ und $\psi(n)$ unbeschränkt); entsprechend $m_{\varphi\psi}{}^m$ ($x\in\tilde{m}$, sonst wie $c_{\varphi\psi}{}^m$); und schliesslich $c_\chi{}^z$, $c_\chi{}^c$, $c_\chi{}^m$, $m_\chi{}^m$, wo $\chi(m, n)$ statt $\varphi(m)$, $\psi(n)$ steht. Für eine Reihe von Matrixtransformationen $\xi_{mn}'=\sum a_{mnkl}\xi_{kl}$ zwischen diesen Räumen werden genaue Bedingungen angegeben. Ferner werden Transformationen $\zeta_{mn}=\sum a_{mnklst}\xi_{kl}\eta_{st}$ betrachtet und es werden zum Teil genaue, zum Teil hinreichende Bedingungen dafür angegeben, dass eine solche Transformation die oben angegebenen Räume ineinander überführt. Diese Sätze werden angewandt auf Vergleichs- und Verträglichkeitssätze bei Doppelfolgen für Nörlund-Voronoi Verfahren und Rieszsche Mittel. Ferner wird die Frage nach der Summierbarkeit der Produkte von Doppelreihen bei diesen Verfahren untersucht. Es ist im Rahmen eines Referates nicht möglich, auf die sehr zahlreichen Einzelheiten einzugehen. *A. Peyerimhoff* (Marburg)

APPROXIMATIONS AND EXPANSIONS

See also 6872.

6967:

Frey, Tamás. Interpolation on normal point sets. I, II. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 121–148, 287–300. (Hungarian)

Let $\{x_{\nu n}\}$ ($\nu\leqq n$) be a triangular matrix of nodes in $[-1, +1]$; $l_{\nu n}(x)$ $[v_{\nu n}(x)l_{\nu n}{}^2(x)]$ the corresponding fundamental polynomials of Lagrange [resp. Hermite-Fejér] interpolation, and $v_{\nu n}(x)\geqq 0$ for all n and ν, $x\in[-1, 1]$. Then for $x\in[-1+\varepsilon, 1-\varepsilon]$ the estimates

$$\left|\prod_{\nu=1}^{n}(x-x_{\nu n})\right| \leqq c_1(\varepsilon)2^{-n},$$

$$\sum_{\nu=1}^{n}|l_{\nu n}(x)| \leqq c_2(\varepsilon)\log n, \qquad \sum_{|x-x_{\nu n}|>\delta}|l_{\nu n}(x)| \leqq c_3(\varepsilon, \delta),$$

$$\sum_{\nu=1}^{n}|x-x_{\nu n}|l_{\nu n}{}^2(x) \leqq c_4(\varepsilon)n^{-1}\log n,$$

$$\sum_{|x-x_{\nu n}|>\delta} v_{\nu n}(x)l_{\nu n}{}^2(x) \leqq c_5(\varepsilon, \delta)n^{-1}$$

hold, from which assertions concerning the convergence of sequences of Lagrange [resp. Hermite-Fejér] interpolation polynomials are deduced in the usual way.

G. Freud (Budapest)

6968:

Aumann, Georg. Über approximative Nomographie. II. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1959**, 103–109 (1960).

Zu einer gegebenen auf $Q=\{(x, y)\colon 0\leqq x\leqq 1, 0\leqq y\leqq 1\}$ stetigen Funktion $f(x, y)$ soll eine stetige Funktion der Form $g(x, y)=a(x)+b(y)$ konstruiert werden, derart dass $\|f-g\|$ minimal ausfällt, wobei $\|f\|=\max\{|f(x, y)|\colon (x, y)\in Q\}$ definiert ist. Zwei Funktionen $f|Q$ und $f_1|Q$ heissen 'verwandt', wenn es Funktionen $a(x)$ und $b(y)$ gibt, derart dass $f=f_1+a+b$ gilt. Zu f ist also eine Verwandte kleinster Norm zu finden. Verf. bezeichnet eine Funktion $g(x, y)$ als 'x-ausgeglichen', wenn $\min\{g(x, y)\colon 0\leqq x\leqq 1\} = -\max\{g(x, y)\colon 0\leqq x\leqq 1\}$ für $0\leqq y\leqq 1$ ist, als 'y-ausgeglichen' unter einer entsprechenden Bedingung, und als '(x, y)-ausgeglichen', wenn g x- und y-ausgeglichen ist. Jede (x, y)-ausgeglichene Verwandte von f ist eine Lösung des Approximationsproblems. Die zu f gehörige 'Symmetrisierungsfolge' $f_0, f_1, f_2, \cdots$ wird definiert durch $f_0=f$, $f_{2n+1}=(f_{2n})^{(1)}$, $f_{2n+2}=(f_{2n+1})^{(2)}$ mit der allgemeinen Bezeichnung

$$f^{(1)}(x, y) = f(x, y)-\tfrac{1}{2}(\max_{x'} f(x', y)+\min_{x'} f(x', y)),$$

$$f^{(2)}(x, y) = f(x, y)-\tfrac{1}{2}(\max_{y'} f(x, y')+\min_{y'} f(x, y')).$$

Hauptsatz: Für jede stetige Funktion $f|Q$ konvergiert die zugehörige Symmetrisierungsfolge gegen eine ausgegliche Verwandte $\tilde{f}$ von f.—Mit dieser Arbeit überträgt Verf. entsprechende Ergebnisse im diskreten Fall (Matrixproblem) [G. Aumann, dieselben S.-B. **1958**, 137–155; MR **22** #1101] auf den kontinuierlichen Fall. Der hier angegebene Beweis ist auch im diskreten Fall anwendbar und stellt dann einen neuen, kürzeren Beweis für die Konvergenz beim diskreten Problem dar. *J. Schröder* (Madison, Wis.)

6969:

Davydov, N. A. On a property of a class of Stieltjes integrals. Mat. Sb. (N.S.) **48 (90)** (1959), 429–446. (Russian)

Soient $\alpha\colon[0, +\infty)\to R$ (la droite réelle) une fonction croissante telle que $\alpha(0)=0$ et $\lim_{t\to+\infty}\alpha(t)=+\infty$, $A\colon[0, +\infty)\to C$ (plan complexe) une fonction continue, et p un nombre naturel. On définit la fonction $A^{(p)}$ sur $[0, +\infty)$ par $A^{(p)}(x)=[(p-1)!]^{-1}\int_0^x(x-t)^{p-1}A(t)d\alpha(t)$; pour $B(t)\equiv 1$ on définit la fonction correspondante $B^{(p)}$. On pose $\sigma^{*(p)}=A^{(p)}/B^{(p)}$ (σ^* est définie sur $(\omega, +\infty)$, où $\omega=\inf\{t|\alpha(t)>0\}$). Soit $\bar{G}\subset C$ un ensemble fermé et convexe. Pour tout $\varepsilon>0$ soit $\bar{G}_\varepsilon$ un domaine fermé convexe contenant $\bar{G}$, tel que la distance de tout point de la frontière de $\bar{G}_\varepsilon$ à $\bar{G}$ soit $\leqq\varepsilon$. On dit que $\bar{G}\neq C$ est un $(\alpha; p)$ ensemble de la fonction A si pour tout $\varepsilon>0$, il existe deux suites (α_k), (β_k) telles que $\alpha_k\leqq\beta_k<\alpha_{k+1}$ et $\alpha_k\to+\infty$, de sorte que $A(x)\in\bar{G}_\varepsilon$ pour $\alpha_k\leqq x\leqq\beta_k$ ($k=1, 2, \cdots$) et

$$\limsup_{k\to\infty} B^{(p)}(\beta_k)/Q^{(p)}(\alpha_k; h_k)<+\infty,$$

où $h_k=(\beta_k-\alpha_k)/p$ et

$$Q^{(p)}(\alpha_k; h_k) = \sum_{m=0}^{p}(-1)^{p-m}\binom{p}{m}B^{(p)}(\alpha_k+mh_k).$$

Définition analogue pour "le point à l'infini du plan complexe est un $(\alpha; p)$ point de A". Le résultat principal est le suivant: Si $\lim_{x\to\infty}\sigma^{*(p)}(x)=s$ et si $\bar{G}$ est un $(\alpha; p)$ ensemble de A, alors $s\in\bar{G}$; si le point à l'infini est un $(\alpha; p)$ point de A alors $\limsup_{x\to\infty}|\sigma^{*(p)}(x)|=+\infty$. Soit maintenant une fonction $a\colon[0, +\infty)\to C$ localement intégrable pour la mesure de Lebesgue. Pour tout $x\geqq 0$ et $p\geqq 0$ posons: $(*)$ $S(x)=\int_0^x a(t)dt$ et $\sigma^{(p)}(x)=(p/x^p)\int_0^x(x-t)^{p-1}S(t)dt$. On dit que l'intégrale $(*)$ est convergente vers s par la méthode de Cesàro d'ordre p ou par la méthode $(c; p)$ si $\lim_{x\to\infty}\sigma^{(p)}(x)=s$. On dit que l'ensemble $\bar{G}\neq C$ est un (C) ensemble de la fonction S si pour tout $\varepsilon>0$ il existe un $\lambda(\varepsilon)>1$ et deux

suites (α_k), (β_k) telles que $\alpha_k \leq \beta_k < \alpha_{k+1}$ et $\alpha_k/\beta_k \geq \lambda(\varepsilon)$, de sorte que $S(x) \in \bar{G}_\varepsilon$ pour $\alpha_k \leq x \leq \beta_k$ $(k=1, 2, \cdots)$. Définition analogue pour "le point à l'infini du plan complexe est un (C) point de S". Du théorème principale on déduit le théorème suivant : Si l'intégrale $(*)$ est convergente vers s par la méthode $(c; p)$ quel que soit $p \geq 0$ et si $\bar{G}$ est un (C) ensemble de S, alors $s \in \bar{G}$. Si le point à l'infini est un (C) point de S alors $\lim \sup_{x\to\infty} |\sigma^{(p)}(x)| = +\infty$ quel que soit $p \geq 0$. Comme conséquences l'A. déduit plusieurs théorèmes taubériennes, par exemple : si l'intégrale $(*)$ est convergente vers s par la méthode $(c; p)$ quel que soit $p \geq 0$, si (x_k) est une suite de nombres positifs tendant vers $+\infty$ et si $\lim_{k\to\infty}(S(y_k) - S(x_k)) = 0$ pour toute suite (y_k) telle que $1 < y_k/x_k \to 1$, alors $\lim_{k\to\infty} S(x_k) = s$.

N. Dinculeanu (Bucharest)

FOURIER ANALYSIS

See also B7238.

6970:

Men'šov, D. E. Convergence of trigonometric series. Proc. Internat. Congress Math. 1958, pp. 398–406. (Russian) Cambridge Univ. Press, New York, 1960.

A lecture surveying both classical and recent results on convergence and divergence of Fourier series and general trigonometric series, with emphasis on the author's recent contributions.

R. P. Boas, Jr. (Evanston, Ill.)

6971:

Boas, R. P., Jr. Beurling's test for absolute convergence of Fourier series. Bull. Amer. Math. Soc. **66** (1960), 24–27.

Let f and g be even, continuous, of period 2π, and let f be a contraction of g, that is, $|f(x)-f(y)| \leq |g(x)-g(y)|$. Let the Fourier cosine coefficients g_n of g be dominated by the terms γ_n of a convergent series. The following result is contained in a paper by Beurling: if $\gamma_n \downarrow 0$ the Fourier series of f is absolutely convergent [Acta Math. **81** (1949), 225–238; MR **10**, 371]. The present author weakens the monotonicity condition on the γ_n. Special case of his complicated sufficient condition: $\sum n^{1/2}\gamma_n < \infty$.

J. Korevaar (Madison, Wis.)

6972:

Rudin, Walter. Trigonometric series with gaps. J. Math. Mech. **9** (1960), 203–227.

Soit f une fonction sommable sur le cercle $|z|=1$, $\sum_1^\infty a_k \exp(in_k\theta)$ sa série de Fourier; la suite $\{n_k\}$ représente une suite de Szidon si $\sum |a_k| < \infty$. Soit E un ensemble d'entiers; la fonction f sera dite une E-fonction si f est sommable et $\hat{f}(n)=0$ pour tout $n \notin E$, où $\hat{f}(n)$ désigne le coefficient de Fourier de f d'ordre n. Le résultat principal de l'auteur, lequel représente une importante contribution relative à la convergence absolue des séries de Fourier lacunaires, est le théorème suivant : La suite E est une suite de Szidon ($\sum |\hat{f}(n)| \leq B\|f\|_\infty$ pour tout E-polynôme f) s'il existe une constante $\delta > 0$ telle qu'à toute fonction b avec $|b(n)|=1$ pour tout $n \in E$ corresponde une mesure μ sur le cercle $|z|=1$, telle que $|\hat{\mu}(n) - b(n)| < 1-\delta$ $(n \in E)$, $\hat{\mu}(n) = \int_{-\pi}^{\pi} \exp(-in\theta)d\mu(\theta)$ $(n=0, \pm 1, \pm 2, \cdots)$. Utilisant ce théorème, on en déduit une simple démonstration du théorème de Stečkin [Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 385–412; MR **18**, 126], lequel de sa part généralise le théorème classique de Szidon.

Soit $0 < r < s < \infty$; E sera de type (r, s), s'il existe une constante B telle que $\|f\|_s \leq B\|f\|_r$ (dans L^p-norme), et ceci pour tout E-polynôme f. Si $0 < s < \infty$, E sera de type $\Lambda(s)$ $(E \in \Lambda(s))$ si E est de type (r, s) pour un $r < s$. Alors l'auteur montre que toute suite de Szidon est de type $\Lambda(q)$ pour tout $q < \infty$, et que $\|f\|_q \leq Bq^{1/2}\|f\|_2$, pour tout E-polynôme f. Cette estimation est dans un certain sens la meilleure. L'article contient des propriétés structurelles des suites de type $\Lambda(q)$, par exemple, les conditions suffisantes pour que $E \in \Lambda(q)$ $(q=4, 6, 8, \cdots)$, ainsi que certaines propriétés analytiques des suites de type $\Lambda(p)$.

M. Tomić (Belgrade)

6973:

Ul'yanov, P. L. Singular integrals and Fourier series. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 33–42. (Russian)

Im Folgenden bezeichnet f eine stetige, 2π-periodische Funktion. Hardy und Littlewood (1925) gaben ein Beispiel eines f für das $\lim_{h\to 0^+} \int_h^\pi [f(x+t)+f(x-t)-2f(x)]t^{-1}\,dt$ nicht existiert für fast alle x. Kaczmarz (1931) und Mazurkiewicz (1931) bewiesen die Existenz von f, für die dieser Grenzwert für kein x existiert, und die Menge dieser Funktionen ist von zweiter Kategorie in der Menge der stetigen, 2π-periodischen Funktionen.—Hier nun wird ein explizites Beispiel eines f gegeben, für das dieser Grenzwert für kein x existiert, während zugleich die Fourierreihe dieser Funktion gleichmäßig konvergiert auf $[0, 2\pi]$. Die Menge aller f, welche diese beiden Eigenschaften zugleich besitzen, ist von erster Kategorie in der Menge aller stetigen, 2π-periodischen Funktionen. Ferner wird bewiesen: Es existieren zwei wechselseitig konjugierte, stetige, 2π-periodische Funktionen $F(x)$ und $\bar{F}(x)$ derart, daß (1) $\int_0^\pi |F(x+t)-F(x-t)|t^{-1}\,dt = \infty$ für alle x, (2) $\int_0^\pi |\bar{F}(x+t) - \bar{F}(x-t)|t^{-1}\,dt = \infty$ für alle x, (3) der Grenzwert $\lim_{h\to 0} \int_h^\pi [F(x+t)-F(x-t)]t^{-1}\,dt$ existiert für alle x, (4) der Grenzwert $\lim_{h\to 0} \int_h^\pi [\bar{F}(x+t)-\bar{F}(x-t)]t^{-1}\,dt$ existiert für alle x, (5) die Fourierreihen von $F(x)$ und $\bar{F}(x)$ konvergieren gleichmäßig auf $[0, 2\pi]$. Bemerkungen von Interesse schließen sich an beide Sätze an.

G. Goes (Evanston, Ill.)

6974:

Chen, Yung-ming. Integrability theorems of trigonometric series. Arch. Math. **11** (1960), 101–103.

Soit $\varphi(x)$ une fonction positive telle que $0 < A < \varphi(t) \times \varphi(x)[\varphi(tx)]^{-1} < B$ $(A, B$ fixe, $t = 1, \frac{1}{2}, \frac{1}{3}, \cdots, x > 0)$;

$$(*) \quad \int_0^1 x^{k+2}\varphi(x)\,dx < \infty, \quad \int_1^\infty x^k\varphi(x)\,dx < \infty, \quad k \geq 0.$$

Soit $\lambda_n \geq 0$ et $\lambda_0/2 + \sum_1^\infty \lambda_n \cos nx$ convergente vers $f(x)$, et soit pour un entier $j \geq 0$

$$\tfrac{1}{2}\lambda_0 + \sum_1^\infty \lambda_n = \sum_1^\infty n^2\lambda_n = \cdots = \sum_1^\infty n^{2j}\lambda_n = 0;$$

alors, avec $k = 2j$ dans $(*)$, $\varphi(x)f(x) \in L(0, 2\pi)$ implique la convergence de $\sum n^{-1}\varphi(n^{-1})\lambda_n$, et inversement. Le résultat analogue a lieu pour la série de sinus. Ces théorèmes généralisent les résultats de J. M. González-Fernández [Proc. Amer. Math. Soc. **9** (1958), 315–319; MR **20** #204], où $\varphi(x) = x^{-\gamma}$. La méthode de démonstration ressemble à celle de la note citée.

M. Tomić (Belgrade)

6975:

Korovkin, P. P. Asymptotic properties of positive methods for summation of Fourier series. Uspehi Mat. Nauk **15** (1960), no. 1 (91), 207–212. (Russian)

Es sei $\rho_k^{(n)}=A_{k,n}/A_n$, $A_n=\sum_{s=0}^{n}\varphi^2(s/n)\neq 0$, $A_{k,n}=\sum_{s=0}^{n-k}\varphi(s/n)\varphi([s+k]/n)$. Für mit 2π periodische Funktionen $f(x)$ wird die trigonometrische Summe $L_{n,\varphi}(f,x)=a_0/2+\sum_{k=1}^{n}\rho_k^{(n)}(a_k\cos kx+b_k\sin kx)$ (a_k und b_k Fourierkoeffizienten von f) gebildet und die Güte der Approximation von f durch $L_{n,\varphi}(f,x)$ untersucht. Ist $f(x)\in C(0,2\pi)$, $\varphi(x)$ R-integrierbar und $\int_0^1\varphi^2(x)dx>0$, so gilt $L_{n,\varphi}(f,x)\to f(x)$ gleichmäßig. Ist

$$c_n=\sup_{f\in Z}\|L_{n,\varphi}(f,x)-f(x)\|,$$

wo Z die Klasse der f mit $|f(x+h)+f(x-h)-2f(x)|<2|h|$ bedeutet, so ist $c_n\geq c\log n/n+O(1/n)$, $c>0$ falls $\varphi^2(x)+\varphi^2(1-x)\geq\lambda>0$, $0\leq x\leq\delta$ ist. Unter weiteren Voraussetzungen über $\varphi(x)$ (typischer Fall $\varphi(x)\equiv 1$, was Fejérsche Mittel ergibt) wird c_n asymptotisch abgeschätzt. Eine weitere asymptotische Abschätzung bezieht sich auf $\varphi(1)=\varphi(0)=0$ (typischer Fall $\varphi(x)=1-2|x-\frac{1}{2}|$, was Jacksonsche Mittel ergibt). Der Verf. bemerkt, dass diese Überlegungen auch auf Approximation durch Ausdrücke der Form L_{n,φ_n} ausgedehnt werden können.

A. Peyerimhoff (Marburg)

6976:

Mikolás, Miklós. Sur la sommation des séries de Fourier au moyen de l'intégration d'ordre fractionnaire. C. R. Acad. Sci. Paris **251** (1960), 837–839.

Das Dirichlet-Verfahren D, erklärt durch $D\text{-}\sum_{n=0}^{\infty}u_n=u_0+\lim_{t\to 0+}\sum_{n=1}^{\infty}u_n\exp(-t\log n)$, wird auf die Fourierreihe $(*)$ $\sum_{n=0}^{\infty}(a_n\cos 2\pi nx+b_n\sin 2\pi nx)$ einer Funktion $f(x)\in L(0,1)$ angewendet. (1) Die Reihe $(*)$ ist fast überall D-summierbar. (2) Ist $f(x)$ beschränkt, so ist $(*)$ genau dann summierbar zum Wert $f\langle\!\langle x\rangle\!\rangle=\lim_{\theta\to 0+}\theta\times\int_0^\delta\frac{1}{2}[f(x+t)+f(x-t)]t^{\theta-1}dt$ ($\delta>0$, beliebig), wenn dieser limes existiert. (3) In regulären Punkten ist $f\langle\!\langle x\rangle\!\rangle=\frac{1}{2}[f(x+0)+f(x-0)]$. (4) Die Summierbarkeit ist in jedem Stetigkeitsintervall gleichmäßig. (5) Entsprechende Ergebnisse werden für ein neues Verfahren $W_{\pm}$ der gebrochenen Integration angegeben, erklärt durch

$$c_0+\lim_{\theta\to 0+}\sum_{-\infty}^{+\infty}{}' c_n(\pm 2\pi ni)^{-\theta}\exp(2\pi nix).$$

D. Gaier (Pasadena, Calif.)

6977:

Srivastava, Pramila. On the abscissa of absolute summability of a Dirichlet series. Indian J. Math. **1**, 77–86 (1959).

Eine Reihe $\sum a_n$ heisst (R,λ,k)-summierbar [bzw. $|R,\lambda,k|$-summierbar], wenn mit $A_\lambda^k(w)=\sum_{\lambda_n<w}(w-\lambda_n)^k a_n$ ($k\geq 0$) der Grenzwert $\lim_{w\to\infty}w^{-k}A_\lambda^k(w)$ existiert [bzw. $w^{-k}A_\lambda^k(w)$ in $(1,\infty)$ schwankungsbeschränkt ist]. Für die Dirichletreihe $\sum a_n\exp[-\lambda_n s]$ seien σ_k und $\bar{\sigma}_k$ die Abszissen der (R,λ,k)- und $|R,\lambda,k|$-Summierbarkeit. Dann ist nach M. Riesz [Acta Lit. Sci. Szeged **1** (1923), 114–128] σ_k eine konvexe Funktion von k:

$$\sigma_p\leq\frac{\sigma_r(p-k)+\sigma_k(r-p)}{r-k},\quad 0<k\leq p\leq r,$$

sofern $\sigma_k>-\infty$ ist. Der Verf. beweist dieses Resultat für $\bar{\sigma}_k$ an Stelle von σ_k.

D. Gaier (Pasadena, Calif.)

6978:

Goes, Günther. Identische Multiplikatorenklassen und C_k-Basen in C_k-komplementären Fourierkoeffizientenräumen. Math. Nachr. **21** (1960), 150–159.

Pour la notation voir les précédents articles de l'auteur dans Math. Z. **70** (1959), 343–371; Math. Ann. **137** (1959), 371–384 [MR **21** #3711, #7392].

Soit P l'ensemble des polynômes trigonométriques et P_∞ l'ensemble des séries trigonométriques formelles $\mathring{f}=(a_j,b_j)=\sum(a_j\cos jt+b_j\sin jt)$, $E\subset P_\infty$ et E^{k*} ($0\leq k\leq 1$) C_k-complémentaire l'espace de E. L'espace normé $E\subset P_\infty$ admet une norme translative invariante si pour tout $\mathring{f}\in E$ et $x\in[0,2\pi]$ on ait $\|\mathring{f}\|_E=\|\mathring{f}(x+t)\|_E$. Désignons par (E,E_1) la classe des facteurs $\lambda=\{\lambda_j\}$ tels que pour tout $\mathring{f}=(a_j,b_j)\in E$ on a toujours $\mathring{T}\mathring{f}=(\lambda_j a_j,\lambda_j b_j)\in E_1$, où $E,E_1\subset P_\infty$, alors le théorème général suivant a lieu. Si E et E_1 sont des espaces du type $\mathrm{BK}\subset P_\infty$ admettant tous les deux normes translatives invariantes, on a $(E,E_1^{k*})=(E,(E_1^{k*})_{kN})$ ($0\leq k\leq 1$), où E_{kN} désigne le sous-ensemble de E dans lequel le système trigonométrique orthonormale forme une C_k-base. Ce théorème contient les critères connus relatifs aux espaces L_p ($1\leq p\leq\infty$), V, A, L_Φ, L_p^*.

M. Tomić (Belgrade)

6979:

Rudin, Walter. Some theorems on Fourier coefficients. Proc. Amer. Math. Soc. **10** (1959), 855–859.

L'auteur construit une suite $\{\varepsilon_n\}$ ($n=1,2,\cdots,\varepsilon_n=\pm 1$) telle que, pour tout N et tout θ, $|\sum_1^N\varepsilon_n e^{in\theta}|<5N^{1/2}$, et donne des applications du type suivant: soit $F(z)$ une fonction paire définie dans le plan des z; une condition nécessaire et suffisante pour que $\sum_{-\infty}^{\infty}F(c_n)e^{in\theta}$ soit la série de Fourier d'une fonction continue, dès qu'il en est ainsi pour $\sum_{-\infty}^{\infty}c_n e^{in\theta}$, est que $F(z)=O(|z|^2)$ quand $z\to 0$. L'auteur signale la priorité de H. S. Shapiro pour la construction de $\{\varepsilon_n\}$.

J.-P. Kahane (Montpellier)

6980:

Helson, Henry; Kahane, Jean-Pierre; Katznelson, Yitzhak; Rudin, Walter. The functions which operate on Fourier transforms. Acta Math. **102** (1959), 135–157.

F étant défini sur un ensemble E du plan complexe, on dit que F opère dans l'algèbre (de fonctions) R si $F(f)\in R$ pour toute $f\in R$ dont les valeurs sont dans E. On suppose F défini sur $[-1,1]$ et $F(0)=0$. Γ étant le groupe dual du groupe G, tous les deux supposés infinis, localement compacts, abéliens, on désigne par $A(\Gamma)$ et $B(\Gamma)$ les algèbres de toutes les transformées de Fourier et celles de Fourier-Stieltjes, respectivement, sur Γ. Voici les résultats principaux: Si Γ est discret et si F opère dans $A(\Gamma)$, F est analytique dans un voisinage de l'origine. Si Γ n'est pas discret et si F opère dans $A(\Gamma)$, F est analytique sur I. Si Γ n'est pas compact, et si F opère dans $B(\Gamma)$, F est une restriction d'une fonction entière. [Voir les travaux suivants, dont les rèsultats cités sont des généralisations, et dont quelques-uns sont démontrés ici: Katznelson, C. R. Acad. Sci. Paris **247** (1958), 404–406; MR **20** #4154; Helson et Kahane, ibid. **247** (1958), 626–628; MR **20** #4737; Kahane et Rudin ibid. **247** (1958), 773–775; MR **21** #1488.]

S. Mandelbrojt (Paris)

6981:

Cooper, J. L. B. Positive definite functions of a real variable. Proc. London Math. Soc. (3), **10** (1960), 53–66.

The author defines positive-definiteness not by the discrete condition $\sum c_r\bar{c}_s f(x_r-x_s)\geqq 0$ but by the integrated condition

$$\iint_{E_s} f(x-y)\psi(x)\overline{\psi(y)}\,dxdy \geqq 0.$$

Whenever this was done before the scope and meaning of the double integral was so drawn that the same class of functions resulted, essentially. But the author obtains classes properly more general and his main case is the following. Denote by $L^2(F)$ the class of functions of bounded support in $(-\infty,\infty)$ and belonging to $L^2(-\infty,\infty)$, and demand that $f(x)$ shall be such that the double integral shall exist Lebesgue for each $\psi\in L^2(F)$ and be $\geqq 0$. Then there is a monotonely nondecreasing $\rho(u)$ in $-\infty<u<\infty$ with $\rho(u)=o(u)$ as $u\to\pm\infty$, such that for almost all x,

$$2\pi f(x) = \int_{-\infty}^{\infty} e^{-iux}\,d\rho(u)$$

in the $(C, 1)$ sense, and that

$$\iint f(x-y)\varphi(x)\overline{\psi(y)}\,dxdy = \int \Phi(u)\overline{\Psi(u)}\,d\rho(u)$$

for any $\varphi,\psi\in L^2(F)$ with Φ, Ψ being their Fourier transforms.

Also, assertions are made in which, in the definition of the double integral, one replaces $L^2(F)$ by a corresponding class $L^p(F)$ with $1\leqq p<\infty$. *S. Bochner* (Princeton, N.J.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

6982:

Pennington, W. B. Widder's inversion formula for the Lambert transform. Duke Math. J. **27** (1960), 561–568.

If $F(s)=\int_0^\infty ta(t)(e^{st}-1)^{-1}dt$, where the integral is convergent for some $s_0>0$, then for $x>0$, $\rho\geqq 0$,

$$\int_0^x (x-t)^\rho a(t)\,dt = \sum_{n=1}^{\infty}\mu(n)n^\rho \lim_{k\to\infty}\frac{(-1)^k}{k!}\left(\frac{kn}{x}\right)^{k+1} \times\left[\left(\frac{d}{ds}\right)^k\frac{\Gamma(\rho+1)F(s)}{s^{\rho+1}}\right]_{s=kn/x}.$$

Differentiation of this formula for $\rho=0$ with respect to x gives an expression for $xa(x)$ at all points of continuity of $a(x)$. This is related to inversion formulas by Widder [Math. Mag. **23** (1950), 171–182; MR **12**, 175]. The author's results do not include Widder's formulae and Widder's do not contain the author's. *W. H. J. Fuchs* (Ithaca, N.Y.)

6983:

Koizumi, Sumiyuki. On the Hilbert transform. I. J. Fac. Sci. Hokkaido Univ. Ser. I **14**, 153–224 (1959).

This paper contains the detailed arguments of the sequence of six papers in Proc. Japan Acad. **34** (1958), 193–198, 235–240, 594–598, 653–656; **35** (1959), 1–6, 323–328; MR **20** #5402, 5403; **21** #2876, 5867, 5868; and following review] and some further results.

In chapter 1 is discussed an extension of the Marcinkiewicz-Zygmund results on interpolation to two mappings, simultaneously effected by a quasi-linear operation, with further mappings. In chapter 2, the Hilbert transform $\tilde{f}(x)=\pi^{-1}$ P.V. $\int_{-\infty}^{\infty} f(t)(x-t)^{-1}dt$ is dealt with in the space $L_\mu{}^p$ ($\equiv L^p$ if $\alpha=0$), with norm

$$\|f(t)\|_{p,\mu} = \left(\int_{-\infty}^{\infty}|f(t)|^p d\mu\right)^{1/p};\quad d\mu=\frac{dt}{1+|t|^\alpha},\quad 0\leqq\alpha<1.$$

If $f\in L_\mu{}^p$, $p\geqq 1$, then $\tilde{f}(x)$ exists almost everywhere. If $p>1$, $\|\tilde{f}\|_{p,\mu}\leqq A\|f\|_{p,\mu}$ $(A=A(p,\mu))$; if $p=1$, the operation is of weak type $L_\mu{}^1\to L_\mu{}^1$, which is discussed in detail. Also cases like $\int_{-\infty}^{\infty}|f|\log^+|(1+t^2)f|d\mu<\infty$ are dealt with. Results are applied to Dirichlet's singular integral. In chapter 3, the class $\mathfrak{H}_\mu{}^p$ is discussed consisting of the functions $f(z)$ analytic for $y>0$ $(z=x+iy)$, with $\|f(x+iy)\|_{p,\mu}\leqq M_{p,a}$ uniformly for $y>0$. Well-known properties of the Hille-Tamarkin class $\mathfrak{H}^p$ ($\equiv\mathfrak{H}_\mu{}^p$ when $\alpha=0$) are generalised to deduce theorem 25: If $g\in L_\mu{}^p$, $p>1$, or both g and $\tilde{g}\in L_\mu{}^1$, then the transform of $\tilde{g}(x)$ equals $-g(x)$.

In chapter 4 a generalised transform $\tilde{g}_1(x)$ is discussed,

$$\text{(1.01)}\qquad \frac{\tilde{g}_1(x)}{x+i}=\frac{1}{\pi}\,\text{P.V.}\int_{-\infty}^{\infty}\frac{g(t)}{t+i}\frac{dt}{x-t},$$

where $g(t)\,(t+i)^{-1}\in L^2$. So are the integrals of the Cauchy and Poisson types, associated with g and $\tilde{g}$. {A number of the results, however, could certainly be reduced to classical results by suitable substitutions; e.g., in (13.21), we may set $g(x)(x-i)^{-1}=g_0(x)\in L^2$, $h(x)(x+i)^{-1}=h_{00}(x)\in L^2$.} In chapter 5, a generalised harmonic analysis of the Hilbert transform is presented. E.g., a modification $s^f(u)$, due to N. Wiener, of the Fourier transform of $f(t)$ is considered, and his Tauberian theorem used to prove the following: if $g(x)$ is real-valued, if $\lim\,(2T)^{-1}\int_{-T}^{T}|g(t)|^2dt$ $(T\to\infty)$ exists, $\int_{-\varepsilon}^{\varepsilon}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2du=o(\varepsilon)$, and if for some constant $a^g=a(g)$

$$\int_0^{2\varepsilon}\Big|\underset{B\to\infty}{\text{l.i.m.}}\int_{-B}^{B} g(t)(t+i)^{-1}e^{-iut}\,dt-\pi a^g\Big|^2du = o(\varepsilon)\text{ as }\varepsilon\to 0,$$

then $\tilde{g}_1(x)$, as defined by (1.01), exists and (correcting an erroneously printed equation)

$$\text{(16.02)}\qquad \lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}|\tilde{g}_1(t)|^2\,dt = \lim\,(2T)^{-1}\int_{-T}^{T}|g(t)|^2\,dt+|a^g|^2.$$

Finally, the results are applied to almost periodic functions. *H. Kober* (Birmingham)

6984:

Koizumi, Sumiyuki. On the singular integrals. VI. Proc. Japan Acad. **35** (1959), 323–328.

A number of theorems are stated concerning a generalised harmonic analysis of the Hilbert transform. The proofs are given in chapter 5 of the paper reviewed above. *H. Kober* (Birmingham)

6985:

Cotlar, Mischa. ★Condiciones de continuidad de operadores potenciales y de Hilbert. [Continuity conditions for

potential and Hilbert operators.] Cursos y Seminarios de Matemática, Fasc. 2. Departamento de Matematica, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Buenos Aires, 1959. iii + 354 pp. (mimeographed) $4.00.

Soient $D \subset E^n$, $D_1 \subset E^m$ (E^n et E^m désignant des espaces euclidiens), μ une mesure sur D et μ_1 une mesure sur D_1. Une application bornée T d'un sous-espace $L_0 \subset L^p(D, \mu)$ dans $L^s(D_1, \mu_1)$ est appelée opérateur de type $(L^p(D, \mu), L^s(D_1, \mu_1))$ sur L_0, ou de type $(L^p(D), L^s(D_1))$ si μ et μ_1 sont les mesures de Lebesgue, ou simplement de type (p, s) si en plus $D = D_1 = E^n$. Le principal but de ce livre est de chercher les types de la transformation de Hilbert H_w et de l'opérateur potentiel H_d: (A) Soit S la sphère unitaire de E^n et w une fonction numérique sur S telle que $\int_S w(t')dt' = 0$; pour $0 < \varepsilon < 1$ considérons la transformée $[H_{w,\varepsilon} f](x) = \int_{\varepsilon<|t|<1/\varepsilon} f(x-t)w(t/|t|)|t|^{-n}dt$ définie pour $f \in L^p(E^n)$, $1 \leqq p \leqq \infty$. Si w vérifie une condition de type Lipschitz et si $1 \leqq p < \infty$, alors pour toute fonction $f \in L^p(E^n)$ il existe $\lim_{\varepsilon\to 0}[H_{w,\varepsilon} f](x)$ presque partout; la limite $[H_w f](x)$ est prise comme définition de l'intégrale $\int_{E^n} f(x-t)w(t/|t|)|t|^{-n}dt$ pour $f \in L^p(E^n)$ et H_w est appelée la transformée de Hilbert; si $1 < p < \infty$, la limite des $H_{w,\varepsilon} f$ existe en moyenne d'ordre p et H_w est de type (p, p). (B) Pour $0 < \varepsilon < 1$ et $0 < d \leqq n$ considérons la transformation $[H_{d,\varepsilon} f](x) = \int_{\varepsilon<|t|<1/\varepsilon} f(x-t)|t|^{d-n}dt$. Si $p = 2n/(n+d)$, alors pour toute fonction $f \in L^p(E^n)$ il existe la limite presque partout de $[H_{d,\varepsilon} f](x)$ quand $\varepsilon \to 0$; la limite $[H_d f](x)$ est prise comme définition de l'intégrale $\int_{E^n} f(x-t)|t|^{d-n}dt$ et H_d est appelé l'opérateur potentiel. Si $1/p - m/(ns) \leqq d/n$, $0 < 1/p < d/(n-m)$, $m \leqq n$, $1/p \neq d/n$ et si D et D_1 sont bornés, alors H_d est de type $(L^p(D), L^s(D_1))$; si, en outre, $1/p - m/(ns) < d/n$, alors H_d est compact; si $1/p - m/(ns) = d/n < 1/p < d/(n-m)$, $1/p < 1$, alors H_d est de type $(L^p(E^n), L^s(E^m))$.

Le livre est divisé en quatre chapitres: Le type des opérateurs; Critères généraux de type; Extension de la notion de type; Propriétés des opérateurs H_d et H_w.

Le premier chapitre est de nature introductoire. On donne la définition du type d'un opérateur. Comme exemples, on montre que: (1) la transformée de Fourier $Tf = f^\wedge$ et son inverse sont de type $(1, \infty)$ et $(2, 2)$; (2) si $k \in L^p(E^n)$ l'opérateur de convolution $Tf = f * k$ est de type (p, p), $1 \leqq p \leqq \infty$; (3) si m est mesurable et bornée sur E^n, l'opérateur multiplicateur T défini par $(Tf)^\wedge = mf^\wedge$ est de type $(2, 2)$. On considère ensuite l'intégrale singulière $(Tf)(y) = F(y) = \int_{E^n} K(x, y)f(x)d\mu(x)$. Pour donner un sens à cette intégrale, et pour définir le type de T, on peut utiliser deux méthodes: (a) on cherche un certain espace L_0 dense dans tout L^p (d'habitude, L_0 est l'espace des fonctions étagées) tel que l'integrale soit définie pour $f \in L_0$; si l'on montre que T est de type (p, s) sur L_0, on le prolonge par continuité à un opérateur de type (p, s) sur L^p, et pour $f \in L^p$ on prend $(Tf)(y)$ comme valeur de l'intégrale précédente. (b) On cherche une suite (K_N) convergente vers K, telle que les opérateurs $(T_N f)(y) = F_N(Y) = \int_{E^n} K_N(x, y)f(x)d\mu(x)$ soient de type (p, s) et que la suite $(\|T_N\|)$ soit bornée; s'il existe la limite (ponctuelle ou en moyenne d'ordre s) $\lim_{N\to\infty} T_N f = Tf$, alors T est de type (p, s), et pour $f \in L^p$ on prend $(Tf)(y)$ comme valeur de l'intégrale précédente. On applique ces deux méthodes pour démontrer le théorème de Plancherel.

Le principal but du second chapitre est de montrer que la transformée de Hilbert H_w est de type $(2, 2)$ (théorème de Lusin pour $n = 1$, de Calderón et Zygmund pour $n > 1$) et que l'opérateur potentiel H_d est de type (p, p^*) pour $p = 2n/(n+d)$, où $1/p + 1/p^* = 1$ (théorème de Hardy et Littlewood pour $n = 1$, de Sobolev et Thorin pour $n > 1$). Pour le faire, on donne d'abord le théorème de convexité pour les fonctions à valeurs opérateurs, duquel on déduit le théorème de convexité de Riesz et Thorin (si T est de type (p_1, s_1) et (p_2, s_2) sur l'espace L_0 des fonctions étagées, alors T est de type (p, s) sur L_0 pour $1/p = t/p_1 + (1-t)/p_2$, $1/s = 1/s_1 + (1-t)/s_2$, quel que soit $0 \leqq t \leqq 1$). En particulier (théorème de Hausdorff, Young et Titchmarsh) on montre que la transformation de Fourier est de type (p, p^*) pour $p \geqq 2$. Puis on donne le théorème des noyaux quasi-orthogonaux. (Une suite $k_i \in L^1(E^n)$ est une suite de noyaux quasi-orthogonaux s'il existe $0 \leqq \varepsilon < 1$ et $M > 0$ tels que $\|k_i * k_{i+j}\|_1 \leqq M^2\varepsilon^j$. Si l'on pose $S_N f = f * \sum_1^N k_i$ et si $\sum_1^\infty k_i^\wedge(u) = h(u)$ ponctuellement et $|\sum_1^N k_i^\wedge(u)| \leqq M$, alors les opérateurs S_N sont de type $(2, 2)$, la suite $(\|S_N\|)$ est bornée et la suite (S_N) converge en moyenne d'ordre 2 vers un opérateur multiplicateur T de type $(2, 2)$, ayant h comme multiplicateur.) De ce théorème on déduit que H_w est de type $(2, 2)$. Puis on donne le théorème des noyaux quasi-orthogonaux en L_p, duquel on déduit que H_d est de type (p, p^*) pour $p = 2n/(n+d)$. On montre aussi que si $0 < d < n$, $m < n < m + 2d$, $p = (m+n)/(m+d)$, H_d est de type $(L^p(E^n), L^{p^*}(E^m))$ et de type $(L^p(E^m), L^{p^*}(E^n))$. À la fin, ce chapitre contient quelques questions complémentaires, comme par exemple, le théorème des opérateurs quasi-orthogonaux dans un espace hilbertien, ou dépendant d'un paramètre continu.

Le principal but du troisième chapitre est de montrer que H_w est de type (p, p) pour $1 < p < \infty$ et que H_d est de type $(L^p(E^n), L^s(E^m))$ pour $1/p - m/(ns) = d/n < 1/p < 1$, $m < n < m + d$ ou $n < m < n + d$. Pour le faire, on définit d'abord le type faible d'un opérateur (T est de type faible $(L^p(D, \mu), L^s(D_1, \mu_1))$, $s < \infty$, s'il existe $M > 0$ tel que $\mu_1(\{x;\ |Tf(x)| > a\}) \leqq (M\|f\|_p a^{-1})^s$ quels que soient $f \in L^p(D, \mu)$ et $a > 0$; T est de type faible (p, ∞) si T est de type (p, ∞)). On montre que H_d est de type faible $(1, n/(n-d))$. On donne le théorème de convexité de Marcinkiewicz (si T est sous-additif et positivement homogène et de type faible (p_1, s_1) et (p_2, s_2), alors T est de type (p, p) pour $p_1 < p < p_2$). On définit aussi la notion de pseudo-type et l'on donne un théorème de convexité pour ce cas-là. Comme application, on déduit que H_w et $H_{w,\varepsilon}$ sont de type faible $(1, 1)$ et de type (p, p), $1 < p < \infty$ (théorème de Riesz et Kolmogoroff pour $n = 1$, de Calderón et Zygmund pour $n > 1$), et de même pour l'opérateur maximal $Mf(x) = \sup_{\varepsilon>0}|H_{w,\varepsilon} f(x)|$ (théorème de Zygmund et Titchmarsh pour $n = 1$, de Calderón et Zygmund pour $n > 1$). On déduit aussi que $\lim_{\varepsilon\to 0} H_{w,\varepsilon} f(x) = H_w f(x)$ ponctuellement pour $f \in L^p$, $1 \leqq p < \infty$. On donne ensuite le théorème de convexité de Marcinkiewicz et Zygmund (si T est de type faible (p_i, s_i), $1 \leqq p_i \leqq s_i \leqq \infty$, $i = 1, 2$, $s_1 \neq s_2$, alors T est de type (p, s) pour $1/p = (1-t)/p_1 + t/p_2$, $1/s = (1-t)/s_1 + t/s_2$ quel que soit $0 < t < 1$). Comme application on déduit que H_d est de type $(L^p(E^n), L^s(E^m))$ pour $1/p - m/(ns) = d/n < 1/p < 1$, $m < n < m + d$ ou $n < m < n + d$, et de type faible $(L^1(E^n), L^s(E^m))$ pour $s = m/(n-d)$. À la fin de ce chapitre on donne quelques questions complémentaires, comme par exemple, les opérateurs de Hilbert ergodiques et les opérateurs doubles de Hilbert.

Le reste des propriétés des opérateurs H_w et H_d annoncées au commencement sont données dans le chapitre IV, c'est-à-dire, que si $D = E^n$ et $1/p - m/(ns) < 1/p < d/(n-m)$,

$1/p<1$, H_d est de type $(L^p(E^n), L^s(E^m))$. Si D et D_1 sont bornés, alors H_d est de type $(L^p(D), L^s(D_1))$ pour $1/p - m/(ns) \leq d/n$, $0<1/p<d/(n-m)$, $1/p \neq d/n$, ou pour $1/p=d/n$ et $s<\infty$, ou pour $1/p=d/(n-m)$ et $1/s<d/(n-m)$; si, en outre, $1/p-m/(ns)<d/n$, alors H_d est compact, et si $1/p<d/n$, H_d est de type compact $(L^p(D), C(D_1))$. À ce but, on donne d'abord le théorème de Young (si $k \in L^r(E^n)$, alors $Tf=f*k$ est de type (p, s) pour $1/p-1/s=1-1/r<1/p<1$) et l'on démontre l'existence de l'unité approximative dans l'algèbre de groupe $L^1(E^n)$. On donne ensuite le théorème d'Arzela concernant les ensembles compacts de $C(E^n)$ et de $L^p(E^n)$, $1 \leq p<\infty$. Puis on considère l'opérateur intégral $[Tf](y)=\int_D K(x, y)f(x)dx$ où $K(x, y)$ est défini sur $D \times D_1$. On donne le théorème de Young généralisé concernant le type de T lorsque le noyau K est assujetti à certaines conditions. On donne aussi des conditions pour que T soit de type compact (p, s). De ce théorème on déduit les résultats sur H_d annoncés plus haut. On définit ensuite les espaces $B_p^{(l)}$ de Beppo-Levi et $W_p^{(l)}$ de Sobolev ($W_p^{(l)}$ est formé des fonctions définies sur D, ayant des dérivées partielles généralisées d'ordre l, ces dérivées appartenant à $L^p(D)$; $B_p^{(l)}$ est le sous-espace de $W_p^{(l)}$ ayant des dérivées partielles habituelles). On donne le théorèmes de plongement de Sobolev (si $p>1$, $1/p<l/n$, alors $W_p^{(l)}(D) \subset C(D)$; si $1/p-m/(ns) \leq l/n<1/p$, $n-pl<m$, alors $W_p^{(l)}(D) \subset L^s(D \cap E^m)$; si, en outre, $1/p-m/(ns) <l/n$, alors l'application identique de $W_p^{(l)}(D)$ dans $L^s(D \cap E^m)$ est compacte). On donne ensuite quelques propriétés de H_w dans l'espace $L^2(E^n)$: si $|w| \log(1+|w|) \in L^1$, alors H_w est un opérateur multiplicateur; si $n=1$, l'unique transformée H de Hilbert est un opérateur unitaire dans L^2; si $n=2$, Hf se développe en série de type Laurent à l'aide d'un opérateur unitaire U, $Hf=\sum c_m U^m f$, $-\infty<m<\infty$. Pour la périodisation H^*f de Hf, on déduit que H^* est de type (p, p) pour $1<p<\infty$ et de type faible $(1, 1)$ (théorème de Calderón et Zygmund). On esquisse ensuite quelques applications aux équations de type elliptiques. À la fin de ce chapitre on donne quelques compléments, concernant la théorie des capacités.

Le livre contient des indications bibliographiques et une liste de 85 ouvrages. Le contenu du livre est exposé d'une manière claire et facile à lire. Il y a une errata, mais le lecteur peut aisément corriger lui-même la plupart des erreurs d'imprimerie. *N. Dinculeanu* (Bucharest)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

6986:

Janoš, Ludvik. Ableitung einer gewissen Ungleichung für die ersten Eigenwerte zweier Randaufgaben. Czechoslovak Math. J. **10** (85) (1960), 68–82. (Russian. German summary)

Soit M et P deux mesures positives de Stieltjes sur $[0, 1]$, $K(x, t)=x(1-t)$ si $0 \leq x \leq t \leq 1$ et $K(x, t)=K(t, x)$ si $0 \leq t \leq x \leq 1$. Considérons le système (1) $\int_0^1 K(x, t)y(t)dM(t) =\alpha z(x)$, $\int_0^1 K(x, t)z(t)dP(t)=\alpha y(x)$ et les équations (2) $\int_0^1 K(x, t)\varphi(t)dM(t)=\beta\varphi(x)$ et (3) $\int_0^1 K(x, t)\psi(t)dP(t) =\gamma\psi(x)$. L'A. démontre que pour les plus grandes valeurs propres α_0 du système (1) et β_0, γ_0 respectivement de l'équation (2) et (3), on a l'inégalité $\alpha_0{}^2 \leq \beta_0\gamma_0$. Le système (1) et les équations (2) et (3) provient de l'intégration du problème aux limites de la barre respectivement de la corde. *N. Dinculeanu* (Bucharest)

6987:

Vekua, N. P. The Cauchy problem for a singular integro-differential equation. Soobšč. Akad. Nauk Gruzin. SSR **22** (1959), 641–648. (Russian)

L'A., utilizzando i risultati conseguiti in un suo lavoro precedente [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **24** (1957), 125–134; MR **20** #960] stabilisce una serie di teoremi concernenti la risoluzione del problema di Cauchy per l'equazione integro-differenziale singolare del tipo

$$(1)\quad \sum_{k=0}^{m}\left[a_k(t_0)\rho^{(k)}(t_0)+\frac{1}{\pi i}\int_L \frac{\Gamma_k(t_0, t)\rho^{(k)}(t)\,dt}{t-t_0}\right]=f(t_0),$$

ove L è un contorno aperto regolare nel piano di variabile complessa $z=x+iy$; $a_k(t_0)$, $\Gamma_k(t_0, t)$, $f(t_0)$ sono funzioni hölderiane note sul contorno L, $\rho^{(k)}(t)=d^k\rho(t)/dt^k$ $(k=0, 1, \cdots, m)$, $\rho^{(0)}(t)=\rho(t)$ è la funzione ricercata e le condizioni di Cauchy per $\rho(t)$ dell'(1) sono $\rho^{(k)}(t^*)=\rho_0^{(k)}$ $(k=0, 1, \cdots, m-1)$, essendovi t^* un determinato punto del contorno L, e $\rho_0^{(k)}$ sono costanti arbitrari prefissate.

Il metodo utilizzato consta nel porre $\rho^{(m)}(t)=\mu(t)$, il che conduce, nelle ipotesi $a_m(t_0)+\Gamma_m(t_0, t_0) \neq 0$ e $a_m(t_0) -\Gamma_m(t_0, t_0) \neq 0$ su L, ad una equazione di Fredholm di seconda specie per la funzione $\mu(t)$ innanzi introdotta; e se poi vi è anche $\Gamma_m(t_0, t_0)=0$, e se l'equazione omogenea associata non ne ha soluzioni $\neq 0$, se ne deduce (teorema I) l'unicità ed effettiva costruzione della soluzione ricercata. Seguono poi i teoremi concernenti l'esistenza delle soluzioni quando l'equazione omogenea già mentovata ne ha n soluzioni linearmente indipendenti.

L'articolo si chiude con le indicazioni concernenti la possibilità di risoluzione del problema di Cauchy per i sistemi di equazioni singolari integro-differenziali come pure nel caso in cui L è costituita da un insieme di contorni chiusi e aperti. *D. Mangeron* (Iaşi)

6988:

Ėbanoidze, T. A. On infinite systems of certain nonlinear regular and singular integral equations. Soobšč. Akad. Nauk Gruzin. SSR **22** (1959), 649–656. (Russian)

L'A., ispirato dall'idea di utilizzare allo studio delle equazioni con operatori oltrechè il noto principio del Tikhonov anche appositamente introdotti spazi topologici con una metrica speciale [cf. B. D. Nikitin, Moskov. Oblast. Pedagog. Inst. Uč. Zap. **57** (1957), 81–98; MR **20** #4751], ne costruisce certi spazi funzionali, notati con $C_{\alpha,R}$ e $L_{\alpha,R}{}^p$ $(p>1)$, e quindi dimostra, per sistemi infiniti di equazioni integrali non lineari

$$(\alpha)\quad y_i(P)=F_i\Big(P, \lambda\int_{G_m} K_1(P, Q, y_1(Q), y_2(Q), \cdots)\,dQ, \\ \lambda\int_{G_m} K_2(P, Q, y_1(Q), y_2(Q), \cdots)\,dQ, \cdots\Big), \\ i=1, 2, \cdots,$$

ove G_m è un dominio ad m dimensioni, λ è un parametro, F_i e K_i sono funzioni date, e y_i $(i=1, 2, \cdots)$ sono funzioni

incognite, come pure per sistemi infiniti di equazioni integrali singolari

(β) $u_i(P) =$

$$F_i\left(P, \lambda\int_{G_m}\frac{\Omega_1(P-Q)K_1(Q, u_1(Q), u_2(Q), \cdots)}{r^m(P, Q)}\,dQ,\right.$$

$$\left.\lambda\int_{G_m}\frac{\Omega_2(P-Q)K_2(Q, u_1(Q), u_2(Q), \cdots)}{r^m(P, Q)}\,dQ, \cdots\right),$$

$$i = 1, 2, \cdots,$$

ove G_m è un insieme arbitrario misurabile dello spazio euclideo ad m dimensioni, r è la distanza tra i punti P e Q dell'insieme G_m, λ è un parametro, F_i, K_i, Ω_i sono funzioni note, u_i sono funzioni incognite e gli integrali sono presi nel senso del valore principale di Cauchy, in certi condizioni, l'unicità e l'esistenza delle soluzioni di tali sistemi negli spazi tuttora mentovati. *D. Mangeron* (Iaşi)

FUNCTIONAL ANALYSIS

See also 6867, 6985.

6989:

Cooper, J. L. B. Functional analysis. Math. Gaz. **43** (1959), 102–109.

Elementary exposition.

6990:

Riedrich, Thomas. Die Stonesche Bedingung und die Äquivalenz der Integrationstheorien von M. H. Stone und N. Bourbaki. Arch. Math. **11** (1960), 206–217.

H. Bauer [Bull. Soc. Math. France **85** (1957), 51–75; MR **19**, 1167] showed that Stone's abstract theory of integration, based on a vector space R of functions, could be brought in a direct way under the Bourbaki theory of integration on locally compact spaces if one adds the condition (S): If $f \in R$, min $(1, f) \in R$, where 1 denotes the function identically equal to 1. The present paper first generalizes the condition (S) to the case of an arbitrary vector lattice with a strong order unit e, and to a Banach lattice with such a unit (termed an F-lattice), replacing 1 by e in the statement of (S). The relation of condition (S) to the maximal ideals in such lattices is brought out. There follows a topologization of the set of maximal ideals of an F-lattice analogous to that of Gel'fand and Šilov [Mat. Sb. (N.S.) **9 (51)** (1941), 25–39; MR **3**, 52] for the case of a Banach algebra. An application of these results gives a simplification of the procedure of Bauer mentioned above, allowing one to remain completely within the framework of Banach lattice theory.

W. R. Transue (Gambier, Ohio)

6991:

Hustad, Otte. On positive and continuous extension of positive functionals defined over dense subspaces. Math. Scand. **7** (1959), 392–404.

Let E be a real locally convex space, and let P be a cone in E which induces a partial ordering on E in the well-known manner. The couple (E, P) is called a [continuous] extension couple (abbreviated [c.] ext. c.) if every positive [continuous] functional on a dense subspace of E has a positive continuous extension to the whole space. In the paper under review various properties related to [c.] ext. c. are established. The main ones are as follows: If P is rich (i.e., for each dense subspace F of E, $P \cap F$ is cofinal in P) and if each positive functional is continuous, then (E, P) is an ext. c. The partial converse holds. Namely, if (E, P) is an ext. c., P is rich; if, in addition, E has a dense subspace which is not a hyperplane, then each positive linear functional is continuous. (E, P) is a c. ext. c. if and only if, for each dense subspace F of E, $P \cap F$ is dense in P. Let $f_1, \cdots, f_n$ be continuous linear functionals on E, and let $P = \{x: f_i(x) \geq 0 \text{ for } i = 1, \cdots, n\}$; then (E, P) is an ext. c.

In the preliminary part of the paper, the existence and non-existence of a dense subspace which is not a hyperplane are discussed in some detail.

I. Namioka (Ithaca, N.Y.)

6992:

Dragilev, M. M. Standard form of basis for the space of analytic functions. Uspehi Mat. Nauk **15** (1960), no. 2 (92), 181–188. (Russian)

Let A be the space of functions regular in $|z| < R$; convergence in A is defined as uniform convergence on every compact subset of $|z| < R$. If T is a continuous linear transformation of A onto A, then the basis $1, z, z^2, \cdots$ of A is transformed into a new basis $f_0(z), f_1(z), \cdots, f_n(z) = Tz^n$. The author proves that every basis of A is of the form $g_n(z) = \lambda_n T z^{k_n}$ $(n = 0, 1, 2, \cdots;$ λ_n constants; $(k_0, k_1, \cdots)$ a permutation of $(0, 1, 2, \cdots))$. The principal step in the proof is the following. Let $\{f_n(z)\}$ be a basis of A and let $\omega_n(r)$ be the number of $f_j(z)$ for which the index of the maximum term in the Taylor series of f_j on $|z| = r$ is $\leq n$. Then $1 \leq \lim\inf_{n\to\infty}\omega_n(r)/n$, $\lim\sup_{n\to\infty}\omega_n(r)/n < \infty$.

W. H. J. Fuchs (Ithaca, N.Y.)

6993:

Práger, Milan. Über ein Konvergenzprinzip im Hilbertschen Raum. Czechoslovak Math. J. **10 (85)** (1960), 271–282. (Russian. German summary)

Let H_i $(1 \leq i \leq k)$ be closed linear subspaces of a Hilbert space H, and $H_0 = \bigcap_{i=1}^k H_i$. Let P_i and P_0 denote the projections of H onto H_i and H_0 respectively. The following results are proved. (I) If $A_n = (P_1P_2 \cdots P_{k-1}P_kP_{k-1} \cdots P_2P_1)^n$, then $\{A_n\}$ converges strongly to P_0. (II) Let $\{j_n\}$ be a sequence of integers such that (i) $1 \leq j_n \leq k$ for each n; (ii) each of the integers $1, 2, \cdots, k$ occurs infinitely often in $\{j_n\}$; (iii) there is a constant K such that $s < t$, $j_s = j_t$ and $j_p \neq j_s$ for $s < p < t$ imply $t - s < K$. If $B_n = P_{j_n} \cdots P_{j_2}P_{j_1}$, then $\{B_n\}$ converges weakly to P_0. (III) If H is finite-dimensional, then for any sequence $\{j_n\}$ with properties (i) and (ii), $\{B_n\}$ converges uniformly to P_0. This last theorem is applied to prove the convergence of the group relaxation method for solving a system of linear equations with a positive definite symmetric coefficient matrix. The case $k = 2$ of (I) is a known result given by J. von Neumann [*Functional operators. II. The geometry of orthogonal spaces*, Princeton Univ. Press, Princeton, N.J., 1950; MR **11**, 599; p. 55] and later independently by N. Wiener [Comment. Math. Helv. **29** (1955), 97–111; MR **16**, 921].

Ky Fan (Detroit, Mich.)

6994:

Gillman, Leonard; Jerison, Meyer. ★Rings of continuous functions. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960. ix+300 pp. $8.75.

Les techniques modernes "abstraites" des mathématiques (algèbre, topologie, structures d'ordre) ont été créées comme moyens d'attaque nouveaux pour les problèmes des mathématiques classiques, et ont surabondamment montré leur fécondité par des succès spectaculaires, notamment dans ces dernières années. Mais ces théories sont aussi devenues pour certains mathématiciens des "fins en soi" et ont été développées sans plus aucun souci d'application à d'autres questions; cela a naturellement conduit à une foule de problèmes nouveaux, souvent fort difficiles, et dont certains n'ont pu être résolus qu'au prix de longs et ingénieux efforts.

L'ouvrage de Gillman et Jerison se rattache à cette tendance: le seul point de contact qu'il pourrait avoir avec d'autres parties de l'Analyse serait la théorie des algèbres normées et de la transformation de Gelfand; mais ces questions ne sont absolument pas mentionnées par les auteurs, et d'ailleurs la structure d'algèbre normée sur l'espace $C^*(X)$ des fonctions continues numériques bornées sur un espace topologique X n'intervient chez eux que d'une façon épisodique et plutôt comme moyen technique. L'objet essentiel du livre est l'étude des structures d'anneau de $C^*(X)$ et de $C(X)$ (espace de toutes les fonctions numériques continues dans X, bornées ou non) et de la façon dont ces structures reflètent les propriétés de la topologie de X.

Le volume débute par une introduction où sont fixées les notations courantes et rappelés les principaux résultats d'algèbre et de topologie utilisés par la suite. Chapitre I: définition et premières propriétés de $C(X)$ et $C^*(X)$; introduction des ensembles de zéros $Z(f)=f^{-1}(0)$ pour $f\in C(X)$; notion de C-immersion [resp. C^*-immersion] d'un sous-espace S de X: cela signifie que toute fonction numérique continue [resp. continue et bornée] dans S admet une extension continue [resp. continue et bornée] dans X. Chapitre II: relations entre les idéaux de $C(X)$ et les bases de filtre formées des $Z(f)$ quand f parcourt un tel idéal: I est un z-idéal si pour $f\in I$ et $Z(g)=Z(f)$ on a $g\in I$; relations entre ces idéaux, les idéaux premiers et les idéaux maximaux. Chapitre III: introduction des espaces complètement réguliers et rappel de leurs principales propriétés; par la suite, tous les espaces considérés sont supposés complètement réguliers. Chapitre IV: introduction de la distinction entre idéaux fixes et idéaux libres dans $C(X)$ (ou $C^*(X)$), les premiers étant les idéaux I tels que l'intersection des $Z(f)$ pour $f\in I$ ne soit pas vide; cas des espaces compacts X, où tous les idéaux sont fixes. Chapitre V: premières propriétés des anneaux quotients $C(X)/I$, notamment dans le cas où I est premier (auquel cas $C(X)/I$ est totalement ordonné); cas où I est maximal, et introduction de la distinction entre les idéaux maximaux réels I, pour lesquels $C(X)/I$ est le corps des nombres réels, et les autres, dits hyper-réels. Chapitre VI: la compactification de Stone-Čech et ses propriétés essentielles. Chapitre VII: détermination des idéaux maximaux de $C(X)$ par le théorème de Gelfand-Kolmogoroff: ils correspondent biunivoquement aux points de la compactification de Stone-Čech βX de X, à un tel point p correspondant l'idéal des $f\in C(X)$ tels que p soit adhérent à $Z(f)$. Chapitre VIII: théorie des espaces "compact-réels" introduits par Hewitt; ce sont les espaces X tels que les idéaux maximaux de $C(X)$ correspondant aux points de $\beta X-X$ soient tous hyper-réels. Tous les espaces de Lindelöf complètement réguliers sont compact-réels; tout sous-espace fermé d'un espace compact-réel et tout produit d'espaces compact-réels est compact-réel; pour tout espace complètement régulier X, le sous-espace vX des $p\in\beta X$ pour lesquels l'idéal maximal correspondant est réel est le plus petit sous-espace compact-réel de βX contenant X et a vis-à-vis des espaces compact-réels les mêmes propriétés "universelles" que βX vis-à-vis des espaces compacts. Chapitre IX: évaluation (qui remonte à Hausdorff) de Card $(\beta X)=$ Card $(\mathfrak{P}(\mathfrak{P}(X)))$ pour tout espace discret infini X; conséquences diverses sur les cardinaux de βX et de $\beta X-X$ pour divers types d'espaces. Chapitre X: étude de la relation canonique entre applications continues $X\to Y$ et homomorphismes d'anneaux $C(Y)\to C(X)$. Chapitre XI: immersions canoniques de βX dans $R^{C^*(X)}$ et de vX dans $R^{C(X)}$, les images étant des sous-espaces fermés et respectivement C^*-immergée et C-immergée. Chapitre XII: caractérisation des espaces discrets compact-réels: ce sont ceux dont le cardinal $\mathfrak{m}$ n'admet pas de mesure ne prenant que les valeurs 0 et 1 et nulle sur les ensembles finis; tous les cardinaux inférieurs au premier cardinal fortement inaccessible ont cette propriété. Minoration du cardinal de C/I pour certains idéaux maximaux I. Chapitre XIII: étude des corps résiduels C/I lorsque I est un idéal maximal hyper-réel: ce sont des corps ordonnés maximaux, dont le degré de transcendance sur R est au moins Card (R); en outre, ils ont la propriété (η_1): pour tout couple de parties dénombrables A, B d'un tel corps, telles que $a\in A$, $b\in B$ entraînent $a<b$, il existe x tel que $a<x<b$ pour $a\in A$ et $b\in B$. Une étude de ces ensembles montre entre autres que si on admet l'hypothèse du continu, tous les corps C/I de cardinal égal à Card (R) sont isomorphes. Chapitre XIV: étude détaillée des anneaux d'intégrité C/I pour un idéal premier I, de leur structure d'ordre et de leurs idéaux premiers, qui forment un ensemble totalement ordonné. On étudie aussi dans ce chapitre deux types spéciaux d'espaces X, ceux pour lesquels tout idéal de type fini de $C(X)$ est principal (F-espaces) et ceux pour lesquels tout idéal premier est maximal (P-espaces); les P-espaces se caractérisent encore, soit comme ceux pour lesquels $C(X)$ est un anneau régulier (au sens de von Neumann), soit comme ceux pour lesquels toute intersection d'une suite décroissante d'ouverts est un ouvert. Chapitre XV: rappel des propriétés classiques des espaces uniformes; caractérisation de βX et vX comme complétés de X pour des structures uniformes convenables. Démonstration du théorème de Shirota: si on se limite aux espaces X dans lequel le cardinal de tout sous-espace discret fermé vérifie la condition du chapitre XII, alors les espaces compact-réels sont exactement ceux admettant une structure uniforme pour lesquels ils sont complets. Chapitre XVI: démonstration du théorème de Stone-Weierstrass; rappels sur la théorie de la dimension; démonstration du théorème de Katětov caractérisant les espaces de dimension $\leq n$ par des propriétés de certains sous-anneaux fermés de $C^*(X)$. Le volume se termine par des notes historiques sur les divers chapitres, une bibliographie et un index très détaillé et complet.

Ce livre servira certainement pendant longtemps d'ouvrage de référence, contenant pratiquement tous les résultats actuellement connus sur ces questions, y compris

d'innombrables exemples et contre-exemples. Le style est excellent par sa clarté et sa concision. On peut seulement regretter que le plan choisi laisse à désirer: la liste qui précède montre qu'on saute constamment d'un sujet à un autre, sans ordre logique apparent; les propriétés se rapportant à une même notion (comme par exemple celle d'idéal premier) sont éparpillées tout au long de l'ouvrage. Enfin, il semble au rapporteur que l'introduction de la notion d'espace uniforme dès le début aurait grandement simplifié et raccourci de nombreux développements, en permettant par exemple d'introduire immédiatement les espaces βX et vX. Il est vrai que visiblement les auteurs n'écrivent pas pour des mathématiciens en quête d'outils de travail et désireux d'arriver au but le plus vite possible, mais pour de vrais "aficionados" de la théorie, disposés à y consacrer tout leur temps et pour qui trois démonstrations différentes du même théorème triple la délectation qu'ils y trouvent. *J. Dieudonné* (Paris)

6995:

Dikanova, Z. T. On some properties of a semi-ordered space of continuous functions on a bicompact. Uspehi Mat. Nauk **14** (1959) no. 6 (90), 165–172. (Russian)

The author solves a problem of Kantorovič, Vulih and Pinsker [same Uspehi **6** (1951), no. 3 (43), 31–98; MR **13**, 361] and gives characterizations of some properties of complete vector lattices. She considers the space $C_\infty(Q)$ of all real-valued continuous functions on a bicompact Q such that each function assumes $+\infty$ and $-\infty$ only on a nowhere dense subset. The closure of any open set in Q is supposed to be open-closed. Conditions are stated under which a complete l-ideal of $C_\infty(Q)$ is of countable type in the sense that every bounded subset of its distinct, pairwise disjoint elements is countable. In the second part necessary and sufficient conditions are given for the convergence in $C_\infty(Q)$ to have one or another property. *L. Fuchs* (Budapest)

6996:

Ehrenpreis, Leon. Theory of infinite derivatives. Amer. J. Math. **81** (1959), 799–845.

L'auteur continue ici ses recherches sur une extension de la théorie des distributions [Ann. of Math. (2) **63** (1956), 129–159; MR **17**, 876], dans l'ordre d'idées de la théorie des fonctions généralisées de Gel'fand et Šilov [Uspehi Mat. Nauk 8 (1953), no. 6 (58), 3–54; Amer. Math. Soc. Transl. (2) **5** (1957), 221–274; MR **15**, 867; **18**, 736].

Soit $a=(a_i)$ une suite de nombres complexes. On dit qu'une fonction complexe f, définie sur l'axe réel R, appartient au domaine de D^a, si $f\in\mathfrak{S}$ (où $\mathfrak{S}$ est l'espace des fonctions indéfiniment dérivables à décroissance rapide) et si, pour tout k, la série $\sum a_i(d^i/dx^i)X^kf$ converge dans $\mathfrak{S}$, X étant la fonction définie par $X(x)\equiv x$; on pose alors $D^af=\sum a_i(d^i/dx^i)f$. On dit que f appartient au domaine de X^a, si $f\in\mathfrak{S}$ et si, pour tout k, $\sum a_iX^if^{(k)}$ converge dans $\mathfrak{S}$; on pose alors $X^af=\sum a_iX^if$. Soit maintenant A une classe de suites de nombres complexes; l'espace K des fonctions f appartenant à X^a [resp. D^a], pour tout $a\in A$, est un sous-espace de $\mathfrak{S}$, caractérisé par certaines conditions de croissance à l'infini [resp. par certaines conditions de régularité], et la transformation de Fourier applique K sur l'espace L des fonctions appartenant au domaine de D^a [resp. de X^a]. On munit ces espaces d'une topologie, d'accord avec l'opérateur considéré; en général, le dual de L [resp. K] est un espace de "fonctions généralisées", qui ne sont pas toutes des distributions.

Pour atteindre plus de généralité, l'auteur considère, au lieu d'une seule classe A de suites de nombres complexes, une suite $\Gamma=C_1, C_2, \cdots, L_1, L_2, \cdots$ de telles classes, et il désigne par $\mathfrak{G}_\Gamma$ (ou par $\mathfrak{G}$) l'espace des fonctions f telles que, pour tout $r\geqq 1$, f appartient au domaine de $\prod_1^r X^{b_i}D^{a_i}$, où $a_i\in C_i$, $b_i\in L_i$, $i=1, 2, \cdots, r$, cet espace étant muni d'une topologie au moyen de certaines semi-normes. D'autre côté, $\hat{\mathfrak{G}}_\Gamma$ (ou $\hat{\mathfrak{G}}$) sera l'espace des fonctions f telles que, pour tout r, f appartient au domaine de $\prod_1^r D^{a_i}X^{b_i}$, muni d'une topologie convenable (on peut d'ailleurs choisir Γ' et Γ'' telles que $\mathfrak{G}_\Gamma=\hat{\mathfrak{G}}_{\Gamma'}$, $\hat{\mathfrak{G}}_\Gamma=\mathfrak{G}_{\Gamma''}$). Cela étant, on démontre que $\mathfrak{G}$ est un espace de Schwartz.

Les éléments de $\mathfrak{G}'$ ou $\hat{\mathfrak{G}}'$ sont appelés "hyperdistributions". Chaque opérateur du type considéré peut se prolonger à $\mathfrak{G}'$ ou à $\hat{\mathfrak{G}}'$ par dualité, et de même pour la transformation de Fourier, qui est un isomorphisme vectoriel-topologique de $\mathfrak{G}'$ sur $\hat{\mathfrak{G}}'$. L'auteur présente plusieurs exemples intéressants d'espaces de hyperdistributions.

Dans une deuxième partie de cet article, l'auteur est amené à considérer des espaces de fonctions holomorphes, dont la topologie, dans certains cas, est celle introduite indépendamment par Köthe [J. Reine Angew. Math. **191** (1953), 30–49; MR **15**, 132], Grothendieck [ibid. **192** (1953), 35–64; MR **15**, 438], da Silva Dias [Thesis, Univ. of São Paulo, 1951; MR **13**, 249], et Van Hove [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38** (1952), 333–351; MR **14**, 287].

{Remarque. Le rapporteur peut affirmer que, quoique ces travaux aient été publiés à des dates différentes, ils ont été écrits à peu près à la même époque, après la lecture de travaux du rapporteur.} *J. Sebastião e Silva* (Lisbon)

6997:

Martineau, André. Supports des fonctionnelles analytiques. C. R. Acad. Sci. Paris **250** (1960), 2666–2668.

Soit $H(V)$ l'espace des fonctions holomorphes sur une variété analytique complexe V (munie de la topologie de la convergence uniforme sur les compacts); l'auteur appelle "fonctionnelles analytiques sur V" les fonctionnelles linéaires continues sur $H(V)$. Étant donné un compact K de V et une fonctionnelle analytique ψ sur V, on dit que ψ est "portée" par K, si ψ est prolongeable, comme fonctionnelle continue, à l'espace $H(K)$, limite inductive des espaces $H(W)$, où W parcourt la famille des voisinages ouverts de K dans V. Si K est minimal pour la relation d'inclusion dans la famille des compacts qui portent ψ, on dit que K est le "support" de ψ.

Soit maintenant E un espace vectoriel complexe de dimension n et soit E^* son dual. Si ψ est une fonctionnelle analytique sur E, on appelle "transformée de Fourier-Borel" de ψ, et on désigne par $F\psi$, la fonction entière (de type exponentiel) définie sur E^* par $(F\psi)(u)=\psi_z(\exp\langle u, z\rangle)$.

L'auteur établit plusieurs propriétés générales des supports, dans le cas où V est une variété de Stein. En particulier, dans le cas où $V=E$, il établit une relation entre le support des fonctionnelles analytiques et l'ordre de croissance de leurs transformées de Fourier-Borel. Comme corollaire, il en déduit, par exemple, que, si D est un "opérateur différentiel d'ordre infini" à coefficients constants et f une fonction analytique au voisinage d'un

compact convexe K de $\mathbf{R}^n$, alors il existe g analytique au voisinage de K telle que $Dg=f$.

{L'auteur appelle "opérateur différentiel" tout opérateur de convolution $\psi *$, où ψ est une fonctionnelle analytique sur E portée par l'origine. Il est cependant à remarquer que, si $E=\mathbf{C}$, les fonctionnelles analytiques sur E (à support quelconque) s'identifient, soit aux germes de fonctions analytiques nulles à l'infini, soit aux ultra-distributions ϕ à support compact; et que, dans ce cas, "tout" opérateur $\phi *$ coïncide avec un opérateur différentiel $\sum_0^\infty c_n D^n$ (où $D=d/dz$), tel que la suite $(n!c_n)^{1/n}$ est bornée, comme nous l'avons montré [Math. Ann. **136** (1958), 58–96; MR **21** #4354]. Ce résultat s'étend d'ailleurs aussitôt à $\mathbf{C}^n$.} *J. Sebastião e Silva* (Lisbon)

6998:

Pisanelli, Domingos. Caratterizzazione della trasformazione di Euler. Bol. Soc. Mat. São Paulo **11** (1956), 107–114 (1959).

Soit $F[y]$ un opérateur linéaire continu, défini dans une région linéaire (A) de l'espace $\mathfrak{S}$ de Fantappiè et à valeurs dans $\mathfrak{S}$. On dit que F est connexe, si sa fonction indicatrice $u(\alpha, z)=F[(\alpha-t)^{-1}, z]$ est définie et holomorphe dans un ouvert connexe $\mathfrak{H}\subset\mathfrak{S}_t\times\mathfrak{S}_z$. L'auteur caractérise les opérateurs E de Euler, comme les opérateurs linéaires continus et connexes E (définis dans (A) et à valeurs dans $\mathfrak{S}$) qui vérifient les conditions de Pincherle: $DE=ED$, $DE'=sE$, où D est l'opérateur de dérivation, $E'=E(ty)-zE(y)$ et s un nombre complexe arbitraire; et il démontre que, dans ce cas, E sera nécessairement de la forme

$$E[y] = k\int_C (t-z)^{-s-1}y(t)\,dt \quad (-\pi < \arg(t-z) < \pi),$$

où k est une constante et C un contour dépendant de y et de z. En général il faut que $\infty\in A$.

J. Sebastião e Silva (Lisbon)

6999:

Pisanelli, Domingos. Sur des conséquences de théorèmes d'approximation de fonctions analytiques. Boll. Un. Mat. Ital. (3) **14** (1959), 301–306.

En s'appuyant sur le théorème de Runge, modifié par Omar Catunda, sur l'approximation uniforme d'une fonction analytique d'une variable par des fonctions rationnelles à coefficients entiers complexes et sur le théorème de l'approximation uniforme d'une fonction analytique de deux variables par des sommes de produits de deux fonctions d'une variable, l'auteur donne de nouvelles démonstrations des deux faits suivants: (α) Tout compact de l'espace $H(K)$ des germes de fonctions holomorphes sur un compact K de la sphère de Riemann admet un système fondamental dénombrable de compacts; (β) La topologie du produit tensoriel projectif de $H(K_1)$ par $H(K_2)$ est identique à la topologie induite par celle de $H(K_1\times K_2)$, d'où l'on déduit $H(K_1\times K_2)\simeq H(K_1)\hat{\otimes} H(K_2)$.

J. Sebastião e Silva (Lisbon)

7000:

Gribanow, Jurij Iwanowitsch. Zur Theorie der Orliczschen Koordinatenräume und der unendlichen Matrizenabbildungen. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **8** (1958/59), 339–349. (Russian, English and French summaries)

Ist $p(t)$ eine nicht abnehmende, rechtsseitig stetige Funktion, $p(0)=0$, $p(t)>0$ für $t>0$, $\lim_{t\to\infty} p(t)=\infty$, dann heißt $M(u)=\int_0^u p(t)\,dt$ $(0\leqq u<\infty)$ eine N-Funktion und $N(v)=\int_0^v q(s)\,ds$ $(0\leqq v<\infty)$ mit $q(s)=\sup_{p(t)\leqq s} t$ die zu $M(u)$ konjugierte N-Funktion. Die Funktionen $M(u)$, $M_1(u)$, $M_2(u)$, im folgenden seien N-Funktionen und N, N_1, N_2 die entsprechenden konjugierten N-Funktionen. Zunächst werden Beziehungen zwischen N-Funktionen untersucht. $M(u)$ heißt vergleichbar mit $M_1(u)$, $M\leqq M_1$, wenn α, $u_0>0$ existieren mit $M(u)\leqq M_1(\alpha u)$ für jedes $u\in[0, u_0]$, äquivalent, wenn $M\leqq M_1$ und $M_1\leqq M$. $M(u)$, $M_1(u)$ und $M_2(u)$ genügen der Δ^3-Bedingung, wenn α, $u_0>0$ existieren, sodaß $M(uv)\leqq M_1(\alpha u)M_2(\alpha v)$, $0\leqq u$, $v\leqq u_0$. $M(u)$ genügt der Δ_2-Bedingung, wenn $\limsup_{u\to 0} M(2u)/M(u)<\infty$. $M(u)$, $M_1(u)$ und $M_2(u)$ genügen der Δ_3-Bedingung, wenn α, u_0 existieren, sodaß $M(uv)\geqq M_1(\alpha u)M_2(\alpha v)$, $0\leqq u$, $v\leqq u_0$. In 9 Sätzen werden Kriterien angegeben bezüglich der Frage, ob gewisse N-Funktionen vergleichbar oder äquivalent sind oder ob sie eine Δ^3-, Δ_2-, Δ_3- u.a. Bedingungen erfüllen. $x=(x_1, x_2, \cdots)\in l^M$, wenn $\rho_M(x)=\sum_{n=1}^\infty M(|x_n|)$ konvergiert. $x\in l_M$, wenn $|\sum_{n=1}^\infty x_n y_n|<\infty$ für alle $y=(y_1, y_2, \cdots)\in l^N$. Mit $\|x\|_M=\sup_{\rho_N(y)\leqq 1}|\sum_{n=1}^\infty x_n y_n|$ wird l_M zu einem Banachraum, genannt Orliczscher Koordinatenraum. $l_M=l^M$ genau dann, wenn $M(u)$ die Δ_2-Bedingung erfüllt. In einem späteren Abschnitt wird gezeigt: Es ist $x\cdot y=(x_1y_1, x_1y_2, x_2y_2, x_2y_1, x_1y_3, x_2y_3, x_3y_3, x_3y_2, x_3y_1, x_1y_4, \cdots)\in l_M$ für jedes $x\in l_{M_1}$ und jedes $y\in l_{M_2}$ genau dann, wenn $M(u)$, $M_1(u)$, $M_2(u)$ der Δ^3-Bedingung genügen, und es existiert dann eine Konstante l, sodaß für jedes $x\in l_{M_1}$ und jedes $y\in l_{M_2}$ $\|x\cdot y\|_M\leqq l\|x\|_{M_1}\cdot\|y\|_{M_2}$. Ist $P_n x=(0, \cdots, 0, x_{n+1}, x_{n+2}, \cdots)$ so ist h_M der Teilraum von l_M für den $x\in l_M$ auch $\lim_{n\to\infty}\|P_n x\|_M=0$ impliziert. $x\in h_M$ genau dann, wenn für jede Zahl $\lambda>0$, $\rho_M(\lambda x)<\infty$. Im letzten Abschnitt der Arbeit werden mehrere Stetigkeit- und Vollstetigkeitsbedingungen für Matrizenabbildungen in Orliczschen Koordinatenräumen angegeben. Dabei ist die Matrizenabbildung $Ax=h$ definiert durch $\sum_{n=1}^\infty a_{mn}x_n=h_m$ $(m=1, 2, 3, \cdots)$. Wenn $M(u)$, $M_1(u)$, $N_2(u)$ der Δ^3-Bedingung genügen und $a=(a_{11}, a_{21}, a_{22}, a_{12}, a_{31}, a_{32}, a_{33}, a_{23}, a_{13}, a_{41}, \cdots)\in l_N$ bzw. $a\in h_N$, dann ist Ax eine stetige bzw. vollstetige Abbildung von l_{M_1} in l_{M_2} und von l_{N_2} in l_{N_1}. Ferner gilt: Die unendliche Matrix $A=(a_{mn})$ definiert eine vollstetige Matrizenabbildung zwischen Orliczschen Koordinatenräumen genau dann, wenn $\lim_{m+n\to\infty} a_{mn}=0$. Zahlreiche Beispiele illustrieren die 25 Sätze. *G. Goes* (Evanston, Ill.)

7001:

Sargent, W. L. C. Some sequence spaces related to the l^p spaces. J. London Math. Soc. **35** (1960), 161–171.

Verf. betrachtet Räume aus Zahlenfolgen $m(\phi)$ und $n(\phi)$, die in folgender Weise erklärt sind. $\{\phi_n\}$ sei eine Folge $0<\phi_1\leqq\phi_n\leqq\phi_{n+1}<\infty$, $(n+1)\phi_n\geqq n\phi_{n+1}$. Eine endliche Menge σ natürlicher Zahlen gehört zu C_s, wenn $\sum_{n\in\sigma} 1\leqq s$. $m(\phi)$ bzw. $n(\phi)$ bestehe aus allen Folgen $x=\{x_n\}$ mit $\|x\|=\sup_{s\geq 1}\sup_{\sigma\in C_s}\phi_s^{-1}\sum_{n\in\sigma}|x_n|<\infty$ bzw. $\|x\|=\sup_{u\in S(x)}\sum_{n=1}^\infty |u_n|(\phi_n-\phi_{n-1})<\infty$ $(\phi_0=0)$, wo $S(x)$ die Menge aller Umordnungen von $\{x_n\}$ ist. Ist

$$\alpha_s = \sup_{\sigma\in C_s}\Big(\sum_{k=1}^\infty \Big|\sum_{n\in\sigma} a_{nk}\Big|^{p'}\Big)^{1/p'}, \quad \mu_t = \sup_{\tau\in C_t}\Big\{\sum_{n=1}^\infty \Big|\sum_{k\in\tau} a_{nk}\Big|^q\Big\}^{1/q},$$

so wird durch $\sum_k a_{nk}x_k$ genau dann l^p in $m(\phi)$ [bzw. $n(\phi)$ in l^q] transformiert, wenn gilt $\sup_{s\geq 1}\alpha_s/\phi_s<\infty$ $(1/p'+1/p=1)$ [bzw. $\sup_{s\geq 1}\mu_s/\phi_s<\infty$]. In diesen Sätzen sind speziell die

Transformationen $l^p \to l^\infty, l \to l^q, l^p \to l, l^\infty \to l^q$ enthalten. Zum Schluss gibt Verf. noch hinreichende bzw. notwendige Bedingungen für Transformationen $l^p \to l^q$ an.

A. Peyerimhoff (Marburg)

7002:

Saphar, Pierre. Sur les sous-espaces invariants d'un opérateur linéaire continu dans un espace vectoriel topologique. C. R. Acad. Sci. Paris **250** (1960), 1165–1166.

Soit E un espace de Banach (complexe), T un opérateur linéaire continu de E et $x \in E$. On dit que x est de type 1 [resp. 2] par rapport à $T-zI$ (où z est un nombre complexe) si l'ensemble des $(T-zI)^n x$ $(n \geqq 0)$ n'est pas [resp. est] topologiquement libre. On étudie le type de $x \in E$ par rapport à $T-zI$ pour z appartenant à l'ensemble résolvant de T.

C. Foiaş (Bucharest)

7003:

Taylor, Angus E. Mittag-Leffler expansions and spectral theory. Pacific J. Math. **10** (1960), 1049–1066.

Let X be a Banach space, A a bounded linear operator of X such that the spectrum $\sigma(A)$ of A consists of 0 and the distinct points $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots, \lambda_n \to 0$; each of the λ_n being a simple pole of $(\lambda I-A)^{-1}$, where I is the identity operator of X. By means of the Mittag-Leffler theorem (applied to vector-valued meromorphic functions) one obtains that $(\lambda I-A)^{-1}$ can be represented in the form $\sum_{n=1}^{\infty} \lambda_n^{\nu_n}\lambda^{-\nu_n}(\lambda-\lambda_n)^{-1}E_n+\Phi(\lambda)$, where $\nu_n \geqq 0$ is an integer and $\Phi(\lambda)$ is an entire function of λ^{-1}. The basic result of the paper concerns the case in which $\nu_n \equiv 1$. It is shown that in this case, $A=B_1+C_1$, $B_1=\sum_{n=1}^{\infty}\lambda_n E_n$, $(\lambda I-B_1)^{-1} =\lambda^{-1}I+\sum_{n=1}^{\infty}[\lambda_n/\lambda(\lambda-\lambda_n)]E_n$ and $\Phi(\lambda)=(\lambda I-C_1)^{-1}$. Special attention is also given to the case $\nu_n \equiv p$.

C. Foiaş (Bucharest)

7004:

Berkson, Earl. Sequel to a paper of A. E. Taylor. Pacific J. Math. **10** (1960), 767–776.

A converse result is given to the one mentioned in the above review, as well as complementary remarks and results on the case $\nu_n \equiv p$ treated in the cited paper of A. Taylor.

C. Foiaş (Bucharest)

7005:

Glimm, James. A Stone-Weierstrass theorem for C^*-algebras. Ann. of Math. (2) **72** (1960), 216–244.

A C^*-algebra is a uniformly closed self-adjoint algebra of operators on a complex Hilbert space. Let $\mathfrak{A}$, $\mathfrak{B}$ be C^*-algebras with $\mathfrak{B} \subseteq \mathfrak{A}$ and $\mathfrak{B}$ containing the identity operator, I. A state f of $\mathfrak{A}$ is a positive linear functional on $\mathfrak{A}$ with $f(I)=1$. A pure state of $\mathfrak{A}$ is an extreme point of the set of states of $\mathfrak{A}$, and the pure state space of $\mathfrak{A}$ is the w^*-closure of the set of pure states of $\mathfrak{A}$. The larger portion of the paper consists of a proof that if $\mathfrak{B}$ separates the pure state space of $\mathfrak{A}$, then $\mathfrak{B}=\mathfrak{A}$. Strong use is made of a similar result of Kaplansky [Trans. Amer. Math. Soc. **70** (1951), 219–255; MR **13**, 48], who assumed that $\mathfrak{A}$ is a CCR-algebra (every irreducible representation consists of completely continuous operators). In the course of the proof, the author obtains a characterization of the w^*-closure of the set of vector states of $\mathfrak{A}$.

Now assume that $\mathfrak{A}$ is, in addition, weakly closed (i.e., a von Neumann algebra). The remainder of the paper consists of a characterization of the pure state space of $\mathfrak{A}$, and a proof that if $\mathfrak{B}$ is w^*-dense in $\mathfrak{A}$ then the pure state space of $\mathfrak{B}$ consists of the set of restrictions to $\mathfrak{B}$ of the elements in the pure state space of $\mathfrak{A}$.

J. A. Schatz (Albuquerque, N.M.)

7006:

Umegaki, Hisaharu. Conditional expectation in an operator algebra. III. Kōdai Math. Sem. Rep. 11 (1959), 51–64.

[For part II see Tôhoku Math. J. (2) 8 (1956), 86–100; MR 19, 872.] Let $B \subset A$, both sigma-finite von Neumann algebras of finite type. Let S_B be the set of all normal states σ of A such that $\sigma(ab)=\sigma(ba)$. Any linear normal idempotent from A onto B is called a 'B-expectation'. Theorem 1: For each $\sigma \in S_B$ there exists a B-expectation ε_σ such that $\sigma(a)=\sigma(a^{\varepsilon_\sigma})$. B is said to be 'sufficient' for a set S_0 of states of A if (1) $S_0 \subset S_B$, and (2) for each $a \in A$ there exists $a' \in B$ such that $a^{\varepsilon_\sigma}=a'$ σ-a.e. In case A is commutative, then the only B-expectation is the usual conditional expectation, so this reduces to the standard notion of 'sufficient statistics'. A corresponding generalization of the Halmos-Savage theorem then is theorem 2: B is sufficient for S_0 if and only if there exists $\pi \in S_B$ such that, for all $\sigma \in S_0$, we have (i) $\sigma \prec \pi$, (ii) $\pi(as_\sigma)=\pi(s_\sigma a)$, s_σ being the support-projection of σ in A, (iii) the Radon-Nikodym derivative of σ with respect to π is affiliated with B. (Observe that if A commutes, then S_B is all states of A, and (iii) becomes vacuous, giving the usual commutative form.) Sample application (theorem 7): The following are equivalent: (i) B is sufficient for S_B, (ii) $B' \cap A \subset B$, (iii) there is a unique B-expectation. This gives a proof of the identity of the B-expectations defined in various ways by von Neumann (his operations $|p|q| \cdots$).

J. Feldman (Berkeley, Calif.)

7007:

Gel'man, A. E. Theorems on implicit abstract functions and problems of stability for operator equations. Dokl. Akad. Nauk SSSR **127** (1959), 945–948. (Russian)

L'auteur énonce plusieurs théorèmes sur les fonctions implicites dans les espaces métriques, en utilisant, au lieu de la dérivée habituelle, une sorte de dérivée métrique, définie par la formule

$$f_y'(x_0, y_0) = \limsup_{\substack{\lambda\to 0\\ \mu\to 0}} \frac{\rho[f(x, y_2), f(x, y_1)]}{\rho(y_1, y_2)}$$

la borne supérieure étant prise pour $\rho(x_0, x)<\lambda$, $\rho(y_1, y_0)<\mu$, $\rho(y_2, y_0)<\mu$, $y_1 \neq y_2$.

Ses théorèmes contiennent comme cas particuliers certains théorèmes sur l'existence et la stabilité des solutions des équations différentielles.

G. Marinescu (Bucharest)

GEOMETRY

See also 6831, 6846.

7008:

★Troisième Colloque de Géométrie Algébrique, tenu à Bruxelles du 17 au 19 décembre 1959. Centre Belge de Recherches Mathématiques. Librairie Universitaire, Louvain; Gauthier-Villars, Paris; 1960. 163 pp. FB 250.

A collection of papers, addresses, and preliminary

reports by the following authors: C. Marchionna-Tibiletti, B. Segre, L. Roth, E. Marchionna, P. Du Val, D. Gallarati, C. F. Manara, L. Nollet, P. Burniat, L. Godeaux, M. Baldassarri. The articles will be reviewed separately.

7009:

Forder, Henry George. ★The calculus of extension. Including examples by Robert William Genese. Chelsea Publishing Co., New York, 1960. xvi+490 pp. $4.95.

This was first published in 1941 [Cambridge Univ. Press, Cambridge, England; MacMillan, Co., New York; MR **3**, 12].

7010:

Manning, Henry P. (Editor). ★The fourth dimension simply explained. Dover Publications, Inc., New York, 1960. 251 pp. $1.35.

This was first published in 1910. It is comprised of a collection of popular explanations of four-dimensional spaces, by various authors (some anonymous), plus an introduction and notes by the editor. The essays were submitted for a prize competition sponsored by the Scientific American magazine. The winner was Graham Denby Fitch.

7011:

Hilbert, David. ★The foundations of geometry. Authorized translation by E. J. Townsend. Reprint edition. The Open Court Publishing Co., La Salle, Ill., 1959. vii+143 pp. $2.50.

A reprint of the original English translation [1902].

7012:

Benz, Walter. Über Möbiusebenen. Jber. Deutsch. Math. Verein. **63**, Abt. 1, 1–27 (1960).

Verf. gibt an Hand von 37 Publikationen, an denen er selbst mit mehreren Arbeiten beteiligt ist, einen überaus inhaltsreichen und gedrängten Bericht über Kreisebenen und Möbiusebenen. Auf die vielen Ergebnisse kann im Rahmen eines Referates nicht eingegangen werden. Inhalt: Kreisebenen. Kreisebenen (K, L) über Körperpaaren $K, L \supset K$. Möbiusebenen. Klassen algebraisch definierter Möbiusebenen. Die Kennzeichnung der Möbiusebenen (K, L), $(L:K)=2$, auf Grund des vollen Satzes von Miquel. Möbiusebenen mit einer Othogonalitätsbeziehung. Möbiusebenen mit eingeschränkter Automorphismengruppe. Möbiusebenen mit einer Winkelvergleichung. Literaturverzeichnis. *R. Moufang* (Frankfurt a.M.)

7013:

Viola, Tullio. Sulla geometria dello spazio limitato. Univ. e Politec. Torino. Rend. Sem. Mat. **18** (1958/59), 23–32.

"Limited space" (observed and measured), with its laws laid down in a system of prime concepts and interrelating postulates, poses even in theory an intricate problem which the author attempts to investigate critically. The "limited space" must be capable of extension (prolungabile) in a certain sense. Of the geometries of such "limited spaces" given in the literature [Hjelmslev, Pasch, Hilbert, Schur], the author singles out the one by F. Schur, *Grundlagen der Geometrie*, 1909. Here point and segment are taken as prime concepts; furthermore, there are 8 postulates, of incidence and order only, connecting them. Its great simplicity facilitates a critical evaluation. The author, for instance, studies the consequences of omitting the term "infinite" in the second postulate of Schur: two points A, B determine uniquely an infinite set of points, the segment AB. *S. R. Struik* (Cambridge, Mass.)

7014:

Viola, Tullio. Sul postulato della completezza lineare dell'Hilbert. Ann. Mat. Pura Appl. (4) **49** (1960), 367–373.

Verf. gibt eine kritische Beleuchtung der Formulierung des Vollständigkeitsaxioms V_2 in der 7. [Teubner, Leipzig, 1930] und der 8. [Teubner, Stuttgart, 1956; MR **18**, 227] Auflage in den *Grundlagen der Geometrie* von David Hilbert und wirft einige offene Fragen auf. Im Anschluss an die Bemerkung auf Seite 30 der 8. Auflage, dass aus dem Vollständigkeitsaxiom der Satz 3 folgt, demzufolge zwischen irgend 2 Punkten einer Geraden noch ein weiterer Punkt liegt, zeigt Verf., dass Satz 3 auch in einer Geometrie gilt, in der die Axiome I–III, V_1 gelten, aber V_2 nicht gilt. Ferner beweist Verf. unter Abwandlung der Hilbertschen Schlussweise [loc. cit., Seite 36–37], dass ein Axiom der linearen Vollständigkeit, in dem die Aufrechterhaltung der Axiome I_{1-2}, II_{1-4}, III_1, IV, V_1 gefordert wird, mit allen übrigen Axiomen verträglich ist.
R. Moufang (Frankfurt a.M.)

7015:

Sibson, R. Cartesian geometry of the triangle and hexagon. Math. Gaz. **44** (1960), 83–94.

7016:

Busulini, Franca. Una dimostrazione del teorema delle mediane in geometria assoluta. Ann. Univ. Ferrara. Sez. VII (N.S.) **8** (1958/59), 17–20. (French summary)

Ein bekannter Satz der ebenen absoluten Geometrie sagt: Ist M Mittelpunkt von AC und M' Mittelpunkt von AC', so werden A, M', C' durch Lote auf der Symmetriegeraden (Mittelsenkrechten) des Punktepaares C', C in A, M, C projiziert. Um in der absoluten Geometrie für ein gegebenes Dreieck ABC zu zeigen, dass die Medianen (Seitenhalbierenden) kopunktal sind, betrachtet Verf. ein Dreieck ABC', dessen Eckpunkt C' nicht in der Ebene des Dreiecks ABC liegt und dessen Medianen kopunktal sind (etwa ABC' gleichseitig), und projiziert ABC' durch Lote auf der Symmetrieebene des Punktepaares C', C in ABC; nach dem genannten Satz gehen dabei Seitenmitten in Seitenmitten und daher Medianen in Medianen über.
F. Bachmann (Kiel)

7017:

Lagrange, René. Sur les systèmes isogonaux de sphères. Ann. Sci. École Norm. Sup. (3) **76** (1959), 305–399.

In a previous paper [J. Math. Pures Appl. **37** (1958), 225–244; MR **21** #4376] the author investigated strictly isogonal systems of hyperspheres, i.e., systems of $n \geqq 2$ unitary hyperspheres in a Euclidean N-space E_N, the angles of each two of them being equal. The present paper deals with a more general case of isogonal systems S_n of n unitary hyperspheres when the cosines of the angles are equal in absolute value only. With every isogonal system

S_n of this kind an n-rowed symmetric matrix can be associated whose diagonal elements are equal to 1 and off-diagonal elements are $\varepsilon_{ij} \cos \alpha$, $\varepsilon_{ij} = \pm 1$. For $n \leqq 7$, the types of isogonal systems (apart from situating the hyperspheres in E_N, from ordering them and from "orienting" any of them) are proved to be uniquely determined by the polynomial in $\cos \alpha$ given as the determinant of the matrix mentioned. In the second part, the author investigates the cases $N=2$ and $N=3$ in detail. It results that for $N=2$ there are four types with $n=5$ and two types with $n=6$, no system existing with $n \geqq 7$. For $N=3$, there are ten types with $n=6$, five types with $n=7$ and three types with $n=8$; no system exists with $n \geqq 9$. *M. Fiedler* (Prague)

7018:

Schörner, Ernst. ★Raumbild-Lehrbuch der Darstellenden Geometrie für Ingenieurschulen. R. Oldenbourg Verlag, Munich, 1960. 153 pp. (with anaglyph viewer). DM 16.00.

This book is not intended for schools of higher learning, but the stereoscopic views are commendable. From this viewpoint the book may be used even by those who do not read German, for visualization of subject matter treated in standard books.

Incidentally, stereoscopic projection slides are available in this country, as mentioned in the preface of *Descriptive geometry* by Earle F. Watts and John T. Rule [Prentice-Hall, New York, 1946]. Spatial pictures, however, for teaching analytic geometry, are also used by O.-H. Keller, *Analytische Geometrie und lineare Algebra* [VEB Deutscher Verlag der Wissenschaften, Berlin, 1957; MR **20** #4221]. *D. Mazkewitsch* (Cincinnati, Ohio)

7019:

Fellman, E. A. Über ein spezielles Problem der Perspektive bei Johann Heinrich Lambert. Verh. Naturf. Ges. Basel **70** (1959), 1–6.

The central perspective of a rectangle is a trapezoid. If the rectangle is divided into congruent rectangles, non-congruent trapezoids are obtained. The author gives an elementary geometric construction of the trapezoids and a computation of the distances between their parallel sides (which decompose the nonforeshortened rectangle into congruent rectangles) when the degree of foreshortening (the parallel sides and their distances) is given. *D. Mazkewitsch* (Cincinnati, Ohio)

7020:

Kárteszi, Ferenc. Rösselsprungserien am unendlichen Schachbrette. Publ. Math. Debrecen **6** (1959), 276–287.

In der vorliegenden Abhandlung gibt der Verfasser eine Darstellungsmethode des vierdimensionalen Punktraumes auf eine geeignete Ebene und verwendet die Methode zum Studium eines kombinatorischgeometrischen Problems, welches sich auf, am Schachbrett durchgeführte, Rösselsprungserien bezieht. *M. Piazzolla-Beloch* (Ferrara)

7021:

Lenhard, H.-Chr. Zerlegung von Tetraedern in Orthogonaltetraeder. Elem. Math. **15** (1960), 106–107.

H. Hadwiger hat in Elem. Math. **11** (1956), 109–110, in "Ungelöste Probleme" Nr. 13 die Frage aufgeworfen, ob im R^n jedes Simplex in Orthogonalsimplexe zerlegt werden kann. Verf. zeigt für den dreidimensionalen Fall, dass dem so ist; die Anzahl der auftretenden Orthogonalsimplexe ist $\leqq 12$. Zum Beweis werden die Simplexe in verschiedene Klassen eingeteilt je nach der Art ihrer Winkel passend unterteilt. *J. J. Burckhardt* (Zürich)

7022:

Sydler, J.-P. Exemples de tétraèdres équivalents (mod. 0). Elem. Math. **15** (1960), 97–99.

Exemples de couples de tétraèdres à différences équivalentes au sens de "Zerlegungs- und Ergänzungsgleichheit" à des cubes, mais qui ne sont pas eux-mêmes équivalents à des cubes. *H. Freudenthal* (New Haven, Conn.)

7023:

Pargeter, A. R. Plaited polyhedra. Math. Gaz. **43** (1959), 88–101.

M. S. Longuet-Higgins [Philos. Trans. Royal Soc. London. Ser. A **246** (1954), 401–450; MR **15**, 980; p. 450] devised a method for making skeletal models of polyhedra, using wire but no solder. The author, with some hints from J. Gorham's *Plaited crystal models* [E. and F. N. Spon, London, 1888], has developed a somewhat analogous method for constructing polyhedral surfaces using paper but no glue. The paper is cut into a special shape, usually involving three or four ramifications. It is then carefully creased, and folded into a plait "such as a small girl uses for her pigtails." Detailed instructions are given for making the convex and non-convex regular polyhedra, also most of the archimedean solids and their reciprocals. When the author refers, on p. 99, to "the (space-filling) truncated cuboctahedron" he presumably means "the (space-filling) truncated octahedron." *H. S. M. Coxeter* (Toronto)

7024:

Brunton, James. The plaited dodecahedron. Math. Gaz. **44** (1960), 12–14.

The author adds a few refinements to the paper by A. R. Pargeter [see the preceding review]. In making a model of a polyhedron all of whose faces are equilateral triangles, the first step is to obtain a regular tessellation of such triangles (the regular tessellation {3, 6}) from which the appropriate "net" can be cut out. The author describes a simple "paper folding" procedure for carrying out this step. He also gives a simpler net than Pargeter's for the regular dodecahedron. *H. S. M. Coxeter* (Toronto)

7025:

Tenca, Luigi. Sul semisferoide. Period. Mat. (4) **37** (1959), 295–300.

7026:

Tenca, Luigi. Elica sferica avente per proiezione sulla base della semisfera una spirale di Archimede. Archimede **12** (1960), 100–101.

L'auteur considère sur une hémisphère de centre O les courbes dont les projections horizontales sur le plan de base sont les spirales d'Archimède de pôle O. Il calcule une aire d'une portion d'hémisphère, et un volume associés à cette figure. *M. Decuyper* (Lille)

7027:

Egerváry, J. Über eine Eingenschaft der Parabel und des Paraboloids. Publ. Math. Debrecen **6** (1959), 269–275.

Pour la déterminations de l'axe de symétrie d'une parabole ou d'un paraboloïde, l'auteur propose, au lieu des formules habituelles qui font intervenir les coordonnées du sommet, une autre méthode à l'aide de laquelle l'axe de symétrie en question est déterminé par la donnée de sa direction et par la connaissance d'un de ses points qui est en général différent du sommet. Cette méthode repose sur les deux théorèmes ci-dessous:

Étant donnée la parabole

$$a_{11}x^2+2a_{12}xy+a_{22}y^2+2a_{13}x+2a_{23}y+a_{33} = 0,$$
$$a_{11}a_{22}-a_{12}^2 = 0, \quad |a_{ik}| \neq 0,$$

considérons les points milieux (qui sont toujours réels) des segments définis par les points d'intersection de cette parabole avec les axes X, Y. Les masses a_{11}, a_{22} placées respectivement à ces points ont pour centre de gravité un point situé sur l'axe de symétrie de la parabole.

Étant donné le paraboloïde

$$a_{11}x^2+a_{22}y^2+a_{33}z^2+2a_{23}yz+2a_{31}zx+2a_{12}xy+2a_{14}x$$
$$+2a_{24}y+2a_{34}z+a_{44} = 0,$$
$$A = |a_{ik}| \neq 0; \quad A_{44} = 0;$$
$$a_{11}a_{22}+a_{22}a_{33}+a_{33}a_{11}-a_{12}^2-a_{23}^2-a_{31}^2 \neq 0,$$

considérons les centres (qui sont toujours réels) des coniques définies comme intersection de ce paraboloïde avec les plans YZ, ZX, XY. Les masses $a_{22}a_{33}-a_{23}^2$, $a_{33}a_{11}-a_{13}^2$, $a_{11}a_{22}-a_{12}^2$ placées respectivement à ces centres ont pour centre de gravité un point situé sur l'axe de symétrie du paraboloïde. *F. Şemin* (Istanbul)

7028:

Villa, Mario. Gli enti iperalgebrici di Corrado Segre. Boll. Un. Mat. Ital. (3) **14** (1959), 214–225.

In questa conferenza, dedicata agli enti iperalgebrici di Corrado Segre, l'A., dopo avere accennato alle varie rappresentazioni reali degli spazi lineari complessi, richiama la nozione di varietà iperalgebrica dell' S_r complesso (introdotta da Corrado Segre) come varietà avente come immagine nell' S_{2r} reale una varietà algebrica. Richiama inoltre la nozione di trasformazione iperalgebrica come trasformazione tra due S_r complessi avente per immagine una trasformazione algebrica (reale) fra gli spazi S_{2r} reali rappresentativi. Tra le trasformazioni iperalgebriche biunivoche (cioé ipercremoniane) si hanno in particolare le trasformazioni pseudocremoniane, che mutano varietà algebriche in varietà iperalgebriche. L'A. accenna inoltre ad ulteriori ricerche cui i lavori di Corrado Segre hanno dato origine. *D. Gallarati* (Genoa)

7029:

Marchionna Tibiletti, Cesarina. Groupe concentré d'intersections de courbes et hypersurfaces algébriques. Applications. 3ième Coll. Géom. Algébrique (Bruxelles, 1959), pp. 9–26. Centre Belge Rech. Math., Louvain, 1960.

The lecture is devoted to the following theorem: Let $F_1=0, \cdots, F_r=0$ be r hypersurfaces of the complex projective space S_r, whose intersection is purely zero-dimensional, precisely i of their points of intersection coinciding in a point P. Let $F_1^\varepsilon, \cdots, F_r^\varepsilon$ be polynomials whose coefficients are analytic functions of a complex variable ε, such that $\mathrm{Lt}_{\varepsilon\to 0} F_t^\varepsilon = F_t$, and such that for $\varepsilon \neq 0$ the common points of $F_1^\varepsilon=0, \cdots, F_r^\varepsilon=0$ are distinct and simple. If S_ε is the set of i linear conditions on the coefficients in a polynomial F in order that the hypersurface $F=0$ may pass through those i common points of $F_1^\varepsilon=0, \cdots, F_r^\varepsilon=0$ which tend to P as limit when $\varepsilon \to 0$, then as $\varepsilon \to 0$, S_ε tends to a well defined limit $S_P{}^i$, consisting of i linearly independent conditions on the coefficients in F, expressible as equations (independent of the degree of F) between the values at P of F and its derivatives of order $<i$ (in terms of a non-homogeneous coordinate system). $S_P{}^i$ is independent of the choice of $F_1^\varepsilon, \cdots, F_r^\varepsilon$ as functions of ε, subject to the above conditions, and is the necessary and sufficient condition for F to belong to the ideal generated by $F_1, \cdots, F_r$ in the ring of formal power series in $(x_1-\alpha_1, \cdots, x_r-\alpha_r)$, where $\alpha_1, \cdots, \alpha_r$ are the non-homogeneous coordinates of P. *P. Du Val* (London)

7030:

Turri, Tullio. Le trasformazioni birazionali dello spazio aventi superficie di punti uniti. Rend. Sem. Fac. Univ. Cagliari **29** (1959), 6–9.

Author's summary: "Le trasformazioni birazionali dello spazio con superficie di punti uniti, quando non siano cicliche, sono trasformazioni di De Jonquiéres con stella di rette unite."

7031:

Du Val, Patrick. Application des idées cristallographiques à l'étude des groupes de transformations crémoniennes. 3 ième Coll. Géom. Algébrique (Bruxelles, 1959), pp. 65–73. Centre Belge Rech. Math., Louvain, 1960.

The author shows that groups of Cremona transformations in the plane and in space are isomorphic to certain crystallographic groups in Minkowski space, and points out that this explains a number of rather surprising geometric correspondences and isomorphisms which have been noted. *G. B. Huff* (Athens, Ga.)

7032:

Herszberg, Jerzy. On a result of Beniamino Segre. Rend. Mat. e Appl. (5) **19** (1960), 168–173.

Let V be a primal of a non-singular irreducible algebraic d-fold U, passing through a point O of U with multiplicity $m \geqq 2$, and consider another primal W of U having at O regular associated behaviour (of index $r<m$) with V [about this notion, cf. B. Segre, Rend. Circ. Mat. Palermo (2) **1** (1952), 373–379; MR **15**, 351]. A dilation of U, possessing as base a non-singular subvariety containing O, and locus of points having all the same multiplicity m for V, transforms U into a non-singular d-fold U'; denote by V', W' the proper transforms of V, W respectively, and by O' a point of V' corresponding to O whose multiplicity on V' is also m. Then (loc. cit.) W' passes through O', having there regular associated behaviour with V'.

Here it is shown that the proper transform C' of the intersection C of V, W does not always contain O', and a criterion is given for C' to pass through O'. *B. Segre* (Rome)

7033:

Turri, Tullio. "Regolarità" delle superficie normali invarianti in un'omografia ciclica. Rend. Sem. Fac. Sci. Univ. Cagliari **29** (1959), 1–5.

Author's summary: "Una superficie normale invariante in un'omografia ciclica è regolare. Perciò ogni superficie irregolare che ammetta una trasformazione ciclica è birazionalmente distinta da una superficie normale invariante in un'omografia ciclica."

7034:

Turri, Tullio. Della mancanza di cubiche gobbe su una superficie cubica trasformata in sè da un'omologia armonica centrale. Rend. Sem. Fac. Sci. Univ. Cagliari **29** (1959), 10–13.

Author's summary: "Si dimostra che sopra una superficie cubica trasformata in sè da un'omologia armonica centrale non esistono cubiche gobbe."

7035:

Spampinato, Nicolò. Sulle falde di Halphen, di ordine *n* e classe *v*, a curva osculatrice alla ligne-origine piana. Ricerche Mat. **7** (1958), 254–264.

A Halphen sheet of order n and class v is one which has locally an equation of the form $F^n=G^{n+v}$ at a point of S_3 where the surfaces F, G are both simple and have distinct tangent planes. This is the author's fifth paper on the subject [same Ricerche **4** (1955), 191–206; **5** (1956), 226–238; **6** (1957), 67–95, 195–204; MR **17**, 1240; **18**, 822; **20** #4552, 4553]. It is shown that the surface $x_3^n=(x_2-x_1^{n+v})^{n+v}$, in which the singular curve is plane, has order and class both equal to $(n+v)^2$.

P. Du Val (London)

7036:

Spampinato, Nicolò. Sui due ordini completi di una superficie dell'S_3 come ente bidimensionale di calotte piane e tridimensionale di E_1. Rend. Accad. Sci. Fis. Mat. Napoli (4) **26** (1959), 202–205. (English summary)

7037:

Spampinato, Nicolò. Sul genere aritmetico di una superficie algebrica. I, II. Rend. Accad. Sci. Fis. Mat. Napoli (4) **26** (1959), 476–477, 478–480. (English summary)

Author's summary. Part I: "Two integers, projective characters of a given algebraic surface are introduced. Their difference is always divisible by 24 and the quotient gives the birational invariant $-p_a-1$, where p_a is the arithmetical genus of the surface."

Part II: "Two integers, projective characters of a given algebraic surface are introduced. Their difference is always divisible by 24 and the quotient gives the birational invariant p_a+1, where p_a is the arithmetical genus of the surface."

7038:

Spampinato, Nicolò. La congruenza $(S_3)_{12}$ dell'S_{15} immagine dell'S_3 quadricomplesso di C. Segre ambiente di ∞^{12} sistemi lineari ∞^2 di rigate cubiche del Cayley. Richerche Mat. **8** (1959), 163–171.

Dans le S_{15} défini par les coordonnées projectives x_j, y_j, z_j, t_j $(j=1, 2, 3, 4)$ on considére la variété $V_{14}{}^n$ d'équation

$$\sigma\sum_j \frac{\partial f}{\partial x_j}t_j+\sigma\sum_{j,l}\frac{\partial^2 f}{\partial x_j\partial x_l}y_jz_l+\sum_{j,l,m}\frac{\partial^3 f}{\partial x_j\partial x_l\partial x_m}y_jy_ly_m=\sigma$$

liée à une forme $f(x_j)$ d'ordre $n\geqq 3$, et les quatre espaces S_3 définis respectivement par les x_j, les y_j, les z_j et les t_j; puis dans chacun d'eux les points $A(x_j=a_j)$, $B'(y_j=b_j)$, $C''(z_j=c_j)$, $D'''(t_j=d_j)$ définissant un espace à trois dimensions: $x_j=h_1a_j$, $y_j=h_2b_j$, $z_j=h_3c_j$, $t_j=h_4d_j$. Cet espace coupe la variété V_{14} selon une surface décomposée en le plan $B'C''D'''$ compté $n-3$ fois et une surface cubique appartenant au systéme linéaire défini par $h_1{}^2h_4=0$, $h_2{}^3=0$, $h_1h_2h_3=0$, systéme indépendant de f. Cette cubique est une réglée de Cayley de droite double $C''D'''$ et telle que le plan $B'C''D'''$ soit tangent tout le long de la droite double. Le point A est simple de plan tangent $AB'C''$, et l'A. détermine les équations paramétriques de la nappe d'origine A. Si l'on fait varier les points A, B', C'', D''' dans leurs plans, le S_3 qu'ils définissent décrit une congruence $(S_3)_{12}$ de S_{15} d'ordre 1; en utilisant les relations $x_j=y_j$, $x_j=z_j$, $x_j=t_j$ on fait correspondre aux quatre points A, B', C'', D''' quatre points A, B, C, D d'un même plan; ce quaterne définit le S_3 quadricomplexe de C. Segre et notre congruence $(S_3)_{12}$ en est la premiére représentation complexe; on définit ainsi un systéme $(V_2)_{14}$ de réglées de Cayley telles que par un point de S_{15} en passe un faisceau; donc celles d'un S_3 $(AB'C''D''')$ se distribuent en ∞^1 faisceaux, celles d'un même faisceau étant en A osculatrices entre elles et à une quadrique bien définie d'un faisceau de quadriques. *B. d'Orgeval* (Dijon)

7039:

Spampinato, Nicolò. Una seconda famiglia di superficie razionali, non rigate, di ordine eguale alla classe. Ricerche Mat. **9** (1960), 136–142.

If n, v are any two mutually prime integers, the surface in S_3 with the equation

$$x_3{}^nx_4{}^{(n+v)^2-n}-(x_2x_4{}^{n+v-1}-x_1{}^{n+v})^{n+v}=0$$

is rational, being parametrisable in the form

$$x_1:x_2:x_3:x_4=\rho:\rho^{n-v}+\sigma^n:\sigma^{n-v}:1.$$

It is shown that its order and class are both equal to $(n-v)^2$. *P. Du Val* (London)

7040:

Godeaux, Lucien. Sur les surfaces de genres arithmétique et géométrique zéro dont le système bicanonique est irréductible. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 362–372.

Let F be a surface with $P_g=P_a=0$, $P_2>2$ (the cases $P_2=1$, 2 have been previously considered by the author [same Acad. (5) **44** (1958), 809–812, 738–739, 942–944; **45** (1959), 52–68, 188–196; Boll. Un. Mat. Ital. (3) **13** (1958), 531–534; MR **20** #6422; **21** #3413, 6369; **22** #1569; **20** #7022], who showed also that $P_2\leqq 10$); then $P_2=\pi$, the virtual genus of the canonical system. It is shown that if the bicanonical system is irreducible, there exists at least one curve Γ such that 2Γ is a bicanonical, $\Gamma+\Gamma'$ a tricanonical, and 2Γ a tetracanonical curve, where $|\Gamma'|$ is adjoint to Γ. Γ, Γ' are a base for curves on

the surface, and $|\Gamma - K|$ (where $|K|$ is the virtual canonical system) the unique torsion.

It is shown also that there exists a surface Φ, with $p_g = p_a = 1$, possessing a quadratic involution without united points (traced on a suitable model of Φ in $S_{2\nu-1}$ by lines meeting two skew $S_{\nu-1}$'s) whose image is the surface F. *P. Du Val* (London)

7041:

Godeaux, Lucien. Sur les surfaces de genres arithmétique et géométrique nuls dont le système bicanonique est irréductible. Acad. Roy. Belg. Bull. Cl. Sci. (5) **46** (1960), 47–52.

Cet article complète un article antérieur [voir l'analyse ci-dessus] dans lequel il est établi que si une surface algébrique F non rationnelle de genres $p_a = p_g = 0$ possède un système bicanonique simple et est de genre linéaire au moins égal à 3 (et au plus égal à 10), elle contient deux systèmes linéaires de courbes $|\Gamma_1|$, $|\Gamma_2|$ tels que ses systèmes tri-, tétra- et pentacanoniques sont respectivement $|C_3| = |\Gamma_1 + \Gamma_2|$, $|C_4| = |\Gamma_1 + \Gamma_2'| = |\Gamma_2 + \Gamma_1'|$, $|C_5| = |\Gamma_1' + \Gamma_2'|$; $|\Gamma_i'|$ = adjoint à $|\Gamma_i|$. L'auteur indique ici que la solution antérieurement signalée, obtenue lorsque $|\Gamma_2|$ est l'adjoint $|\Gamma_1'|$ à $|\Gamma_1|$, n'est pas la seule. Il montre que les systèmes $|\Gamma_1|$ et $|\Gamma_2|$ sont réguliers, que les courbes tricanoniques découpent sur les courbes Γ_1 et Γ_2 des séries linéaires non spéciales, et que Γ_1 et Γ_2 découpent sur une courbe bicanonique irrégulière des séries complètes. L'hypothèse où les systèmes $|\Gamma_1|$ et $|\Gamma_2|$ coïncident est spécialement envisagée, et les conditions pour que cette coïncidence ait lieu sont données.
P. Vincensini (Marseille)

7042:

Godeaux, Lucien. Sur les surfaces de genres nuls possédant des courbes bicanoniques irréductibles. J. Math. Pures Appl. (9) **39** (1960) 221–230.

The paper opens with a brief survey of work done hitherto on algebraic surfaces with genera $p_g = p_a = 0$, $P_2 \neq 0$. It is then shown that if an algebraic surface has genera $p_g = p_a = 0$, $P_2 = 4$ or 5, and the general curve of the bicanonical system is irreducible, there is on it an isolated curve Γ of genus P_2 and virtual degree $P_2 - 1$, such that 2Γ is bicanonical. *P. Du Val* (London)

7043:

Roth, Leonard. On irregular varieties which contain cyclic involutions. Rend. Sem. Mat. Univ. Padova **30** (1960), 149–160.

Da note ricerche di M. De Franchis ed A. Comessatti segue che ogni modello semplice di un piano doppio di irregolarità $q > 0$ contiene un fascio di genere q di curve, e quindi non possiede un modello proprio sulla propria varietà di Picard-Severi. L'Autore presenta alcune estensioni di questo teorema, considerando una V_r $(r \geq 2)$ di irregolarità superficiale $q > 0$, che contenga un'involuzione del second'ordine I_2. Denotando con $g_k(I_2)$ $(k = 1, \cdots, r)$ il numero delle forme differenziali linearmente indipendenti di prima specie e di grado k attaccate a V_r, si hanno i seguenti teoremi: (a) Se $g_1(I_2) = g_2(I_2) = 0$, V_r possiede una congruenza avente irregolarità superficiale q di sottovarietà (di qualche dimensione $s \geq 1$); (b) Se $g_1(I_2) = 0$ e $g_k(I_2) < \binom{q}{k}$, per ogni k pari V_r possiede una congruenza irregolare di irregolarità superficiale $\leq q$. Il lavoro contiene interessanti considerazioni preliminari, ed alcune applicazioni al caso in cui V_r sia un S_r doppio irregolare. Si mette inoltre in evidenza che gli stessi metodi possono riuscire utili anche nel caso in cui V_r contenga un'involuzione ciclica d'ordine $m \geq 3$, superficialmente irregolare. *D. Gallarati* (Genoa)

7044:

Severi, Francesco. Nuove relazioni fra il genere aritmetico d'un'ipersuperficie generica elementare A, tracciata sopra una varietà algebrica V_d e i generi aritmetici delle varietà caratteristiche di $|A|$. Ann. Mat. Pura Appl. (4) **50** (1960), 419–420.

La présente note complète une note parue sous le même titre dans les Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **26** (1959), 3–5 [MR **21** #4953]. On y avait établi que si A est une hypersurface tracée sur la variété algébrique V_d de dimension d et est la variété générique d'un système linéaire $|A|$, si $|B|$ est le système adjoint de A, alors

$$[1+(-1)^{d-1}]P_a{}^d + P_a{}^{d-1} - d - 1 = \sum_{s=1}^{s=d} (-1)^{d-s} P_a{}^{d-s},$$

où $P_a{}^d$, $P_a{}^{d-s}$ sont les genres arithmétiques de V_d, B^s. L'auteur montre qu'on peut écrire encore $d-2$ relations analogues, en utilisant les variétés $A^2, A^3, \cdots, A^{d-1}$ caractéristiques du système linéaire $|A|$.
M. Decuyper (Lille)

7045:

Schreier, Otto; Sperner, Emanuel. ★Projective geometry of n dimensions. Vol. 2 of "Introduction to modern algebra and matrix theory". Chelsea Publishing Co., New York, 1961. 208 pp. $4.95.

For volume 1 (1951), see MR **13**, 5.

7046:

Turri, Tullio. Sulle proiettività e antiproiettività nel campo reale. Rend. Sem. Fac. Sci. Univ. Cagliari **29** (1959), 190–211.

Expository article dealing with matrix representations of collineations and correlations in complex and real n-dimensional projective space.
H. Schwerdtfeger (Montreal)

7047:

Bereis, Rudolf; Brauner, Heinrich. Beiträge zur Theorie des mit einer euklidischen Schraubung verknüpften kubischen Nullsystems. Math. Nachr. **20** (1959), 239–258.

Wenn man bei einer Schraubung jedem Raumpunkt die Bahnschmiegebene der durch ihn gehenden Bahnkurve zuweist, dann entsteht ein kubisches Nullsystem. Die Verf. betrachten, vom konstruktiven Standpunkte aus u.a. das einer geraden Punktreihe entsprechende Gebilde und zwar unter Heranziehung der sogenannten Netzprojektion. Es wird eine Fülle von interessanten geometrischen Einzelkeiten hergeleitet. *O. Bottema* (Delft)

7048:

Fuks, B. A. Stereographic projection in the space of n complex variables and some of its applications. Issledovaniya po sovremennym problemam teorii funkcil kompleksnogo peremennogo, pp. 294–300. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The author generalizes stereographic projection to a mapping of the complex projective space P_n into the Euclidean space $E_{n(n+2)}$ of the real variables $x_{pq}^{(1)}$, $x_{pq}^{(2)}$, t_k ($p, q=1, \cdots, n+1$; $p<q$; $k=1, \cdots, n$). The mapping is given by the relations

$$x_{pq}^{(1)}+ix_{pq}^{(2)} = N^{-1}(2n^{-1}(n+1))^{1/2}z_p\bar{z}_q,$$

$$t_k = N^{-1}((n+1)n^{-1}k^{-1}(k+1)^{-1})^{1/2}\Big(\sum_{s=1}^{k} |z_s|^2-k|z_{k+1}|^2\Big),$$

$$N = |z_1|^2+\cdots+|z_{n+1}|^2.$$

The image is part of the unit sphere and the distance between image points is given by a formula similar to the one which holds for the classical stereographic projection.

H. Tornehave (Copenhagen)

7049:

Niče, Vilim. Der Achsenkomplex der oskulatorischen Kreiszylinder eines Kreises und einige seiner Flächen und Kongruenzen. Rad Jugoslav. Akad. Znan. Umjet. **314**. Odjel Mat. Fiz. Tehn. Nauke 7, 93–109 (1959). (Croatian and German)

Die früher vom Verf. durchgeführten Betrachtungen in der Arbeit veröffentlicht in Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II 4 (1949), 1–10 [MR **11**, 207] werden hier weiter entwickelt. Die Achsenkongruenz der hyperoskulatorischen Kreiszylinder eines Kreises ist ein singuläres Gebilde, während der hier betrachtete Achsenkomplex vom sechsten Grade ist und selbstverständlich rotatorisch. Als eine wichtige Besonderheit dieses Komplexes ist eine Regelfläche vierten Grades V. Art erwähnt. Die Erzeugenden dieser Regelfläche sind die Achsen der oskulatorischen Kreiszylinder eines Kreises C in einem seiner Punkte P, die mit der Berührungsgerade p dieses Kreise im Punkte P den gleichen Winkel φ einschliessen. Diese Regelfläche besitzt ein Paar isotroper Torsalgeraden mit isotropen Torsalebenen. Betrachtet man der Winkel φ als einen veränderlichen Parameter, so entspricht ihm ein Büschel derartiger Regelflächen, deren Striktionslinien vierter Ordnung I. Art sind und eine interessante Fläche sechster Ordnung bilden, die auch das erwähnte, allen diesen Regelflächen gemeinsame isotrope, Geradenpaar enthält.

T. P. Anđelić (Belgrade)

7050:

Crampe, Sibylla. Schliessungssätze in projektiven Ebenen und dichten Teilebenen. Arch. Math. **11** (1960), 136–145.

In a topological projective plane, a subplane is said to be dense if every point of the plane is a limit point of points of the subplane. An open, or constructible, configuration theorem is one in which the configuration of the hypothesis is open, though with the incidence of the conclusion the configuration will be closed. It is proved here that the validity of an open configuration theorem in a dense subplane will imply the validity of the theorem in the plane. But a counterexample shows that having the ternary ring of the subplane archimedean in the ternary ring of the plane is insufficient for making theorems of the subplane also valid for the whole plane.

Marshall Hall, Jr. (Pasadena, Calif.)

7051:

Rodriquez, Gaetano. Un esempio di ovale che non è una quasi-conica. Boll. Un. Mat. Ital. (3) **14** (1959), 500–503.

Verf. gibt einen kurzen Literaturbericht über Ergebnisse von B. Segre [Canad. J. Math. **7** (1955), 414–416; MR **17**, 72], B. Qvist [Ann. Acad. Sci. Fenn. Ser. A. I. Math.-Phys. No. 134 (1952); MR **14**, 1008], H. Lenz [Jber. Deutsch. Math. Verein. **57**, Abt. 1 (1954), 20–31; MR **15**, 893], T. G. Ostrom [Canad. J. Math. **7** (1955), 417–431; MR **17**, 400], A. Wagner [J. London Math. Soc. **33** (1958), 25–33; MR **20** #249], J. André [Math. Z. **62** (1955), 137–160; MR **17**, 73], bezüglich der Klassifikation der Ovale in endlichen desargues'schen bez. nicht-desargues-schen Ebenen; im letzteren Falle ist über Ovale wenig bekannt. Verf. führt ein erstes Beispiel für ein Oval in einer nicht-desargues'schen Ebene an, das nicht aus den absoluten Punkten einer Polarität besteht, indem er in einer Ebene über dem Dicksonschen Fastkörper der Ordnung 9 die Koordinaten von 10 Punkten angibt, die zu je dreien nicht kollinear sind.—Verf. kündigt eine ausführliche Darstellung seiner Überlegungen an.

R. Moufang (Frankfurt a.M.)

7052:

Reiman, István. Geometrische Untersuchung einer quadratischen Kongruenz. Mat. Lapok **10** (1959), 122–126. (Hungarian. Russian and German summaries)

Der Verf. beweist auf geometrischem Wege den Bekannten Satz, wonach die quadratische Kongruenz

$$(1)\quad \sum_{\substack{i=1\\k=1}}^{3} a_{ik}x_ix_k \equiv 0 \pmod p$$

$$(p \text{ prim},\ a_{ik} \equiv a_{ki},\ |a_{ik}| \not\equiv 0 \pmod p)),$$

p^2 Lösungen besitzt. Zum Beweis wird über dem Körper der Restklassen modulo p eine endliche projektive Ebene konstruiert. In dieser Ebene wird, ähnlich wie in der klassischen projektiven Geometrie, ein Begriff der Polarität eingeführt, und ein Punkt wird absolut genannt, wenn er auf seinen polaren Geraden liegt. Das zu einem absoluten Punkt gehörige Elemententripel (x_1, x_2, x_3) befriedigt (1). Indem wir nun die Anzahl der absoluten Punkte berücksichtigen, ergibt sich daraus die Behauptung des Satzes auf einfache Weise. Der Verfasser weist auch darauf hin, dass es sich ähnlich beweisen lässt, dass die Gleichung $\sum_{i,k=1}^{3} a_{ik}x_ix_k = 0$, in der $a_{ik}=a_{ki}$ und Det $|a_{ik}| \neq 0$, p^{2s} Lösungen besitzt über einem endlichen Körper der Ordnung p^s.

L. Gyarmathi (Debrecen)

7053:

Hughes, D. R. Review of some results in collineation groups. Proc. Sympos. Pure Math., Vol. 1, pp. 42–55. American Mathematical Society, Providence, R.I., 1959.

Verf. beschreibt zunächst endliche Translationsebenen und ihre Koordinatenbereiche (Quasikörper oder V-W-Systeme). Im Fall nicht assoziativer, beiderseits distributiver Quasikörper oder Divisionsringe ist die Faktorgruppe

der vollen Kollineationsgruppe nach der von den zentralen Kollineationen erzeugten projektiven Gruppe die sog. Autotopismengruppe des Divisionsrings. Er zählt dann mit Definitionen und einfachsten Eigenschaften die 8 bisher bekannten Typen endlicher projektiver Ebenen auf, die bis auf die von ihm selbst entdeckten Hughes-Ebenen [Hughes, Canad. J. Math. **9** (1957), 378–388; MR **19**, 444] sämtlich Translationsebenen sind, und macht einige Angaben über ihre Kollineationsgruppen. Die Kollineationsgruppen der Hughes-Ebenen wurden von Zappa [Boll. Un. Mat. Ital. (3) **12** (1957), 507–516; MR **19**, 876] und Rosati [ibid. **13** (1958), 505–513; MR **20** #7245] bestimmt. Die Kollineationsgruppen der Ebenen über den von M. Hall [Trans. Amer. Math. Soc. **54** (1943), 229–277; MR **5**, 72] angegebenen Quasikörpern hat Verf. [Amer. J. Math. **81** (1959), 921–938; **82** (1960), 113–119; MR **22** #930, 931] untersucht. Sie sind nie auflösbar. Nichts bekannt ist über die Andréschen Translationsebenen [Math. Z. **60** (1954), 156–186; MR **16**, 64; insbes. 185], soweit sie keinen assoziativen Quasikörper (Fastkörper) als Koordinatenbereich haben. Die Kollineationsgruppen der übrigen bekannten Typen nicht desargues'scher endlicher Translationsebenen sind auflösbar. Verf. vermutet, daß jeder endliche Divisionsring eine auflösbare Autotopismengruppe besitzt. Abschließend erwähnt er eine Reihe allgemeinerer Sätze über Kollineationsgruppen endlicher projektiver Ebenen, deren wichtigster der Satz von Ostrom und Wagner [ibid. **71** (1959), 186–199; MR **22** #1843] ist: Jede endliche Ebene mit einer zweifach transitiven Kollineationsgruppe ist desargues'sch.

H. Salzmann (Frankfurt a.M.)

7054:

Segre, Beniamino. Le geometrie di Galois. Ann. Mat. Pura Appl. (4) **48** (1959), 1–96.

Der ausführliche Bericht behandelt systematisch die algebraischen, geometrischen und arithmetischen Eigenschaften der projektiven Räume $S_{r,q}$ der Dimension r über einem endlichen Körper der Ordnung q hinsichtlich gewisser Punktmengen dieser Räume. Dabei sind zahlreiche Veröffentlichungen berücksichtigt, zu denen Verf. selbst den grössten Teil beigetragen hat. Die Arbeit zerfällt in vier größere Abschnitte:

§ I behandelt die Quadriken in einem Raum über einem Galoisfeld mit rein geometrisch kombinatorischen Methoden, ihre Homographien und ihre Schnitte mit Unterräumen. Hauptziel ist eine vollständige Klassifikation der Quadriken.

In § II wird für ebene algebraische Kurven C_n über einem beliebigen algebraisch abgeschlossenen (endlichen oder unendlichen) Körper der Satz des Menelaos erweitert auf Punktgruppen, die von einer C_n auf den Seiten eines Dreiecks ausgeschnitten werden. Schranken für die Zahl der Punkte auf einer C_n des $S_{2,q}$ werden angegeben.

In § III und § IV werden die als K-Bögen ($r=2$) und K-Kalotten ($r\geq 3$) bezeichneten Punktmengen des $S_{r,q}$ behandelt. Dabei ergeben die Fälle q gerade oder ungerade verschiedene Strukturaussagen. Für $r=2$ gibt § III detaillierte Untersuchungen über den Begriff des Index und Rangs, über vollständige K-Bögen, ihre Aufzählung für kleine Werte von q und die Nichtexistenz gewisser K-Bögen. § IV behandelt für $r\geq 3$ und $2\leq s\leq r$ die Punktmengen $K^s_{r,q}$ des $S_{r,q}$, von denen je $s+1$, aber nicht $s+2$ linear unabhängig sind; sie heißen für $s=r$ K-Bögen $K_{r,q}$ und für $s=2$ K-Kalotten $K^*_{r,q}$. Besonderes Interesse beanspruchen die sogenannten vollständigen $K^s_{r,q}$, für die K bei gegebenen r, q, s mit $2\leq s\leq r$ maximalen Wert hat; sie heißen speziell für $s=r$ Ovale $M_{r,q}$ und für $s=2$ Ovaloide $M^*_{r,q}$; ihre Ordnung wird mit $[M]_{r,q}$ resp. $[M]^*_{r,q}$ bezeichnet. Die Untersuchungen über die zahlreichen geometrischen Eigenschaften der $K^s_{r,q}$ beziehen sich unter anderem auf ihre Tangenten, auf die K-Kalotten, die auf einer Quadrik liegen und für deren K eine obere, von q abhängige Schranke angegeben wird, und auf die nähere Charakterisierung der Ovaloide. Zu der im allgemeinen Fall noch unbeantworteten Frage, für welche K, r, q vollständige $K^*_{r,q}$ existieren, die keine Ovaloide sind, und sie gegebenenfalls zu klassifizieren, gibt Verf. Beiträge für $r=3$, q ungerade bzw. q Potenz einer Primzahl; für $r=3$, q ungerade, gibt er ein notwendiges und hinreichendes Kriterium für K an, damit eine $K^*_{r,q}$, die kein Ovaloid ist, vollständig ist. Als asymptotische Eigenschaften von Kalotten bezeichnet Verf. solche Aussagen für $K^*_{r,q}$, die für große Werte von q gelten. Ist z.B. $K\geq q^2-cq$, wo $c\geq 0$ gegeben und q genügend groß ist, so ist jede aus K Punkten bestehende $K^*_{3,q}$ in einer elliptischen Quadrik enthalten; für die Ordnung $[M]^*_{r,q}$ einer $K^*_{r,q}$ mit maximalem $K=M$ gilt

$$\lim_{q\to\infty}\frac{[M]^*_{r,q}-q^{r-1}}{q^{r-2}}=-\infty.$$

Für $[M]^*_{r,q}$ lassen sich mit Unterscheidung der Fälle $r\equiv 0, 1, 2 \pmod 3$ auch untere Schranken angeben, die von q und den Darstellungsgrößen von r abhängen, wobei in jedem Falle K-Kalotten aus einer Anzahl von Punkten existieren, die der unteren Schranke gleich ist.

R. Moufang (Frankfurt a.M.)

CONVEX SETS AND GEOMETRIC INEQUALITIES

7055:

Zarankiewicz, K. Bisection of plane convex sets by lines. Wiadom. Mat. (2) **2**, 228–234 (1959). (Polish)

Let S be a plane convex set, Fr(S) its boundary, and P and P' points of Fr(S). A segment PP' which divides the area of S into two equal parts is called a diameter of S. Theorem 1: Every bounded plane convex set possesses a point through which three different diameters pass. If there is only one such point O, every line passing through O is a diameter. Theorem 2: A necessary and sufficient condition that the point O be a center of symmetry of a bounded convex plane set is that O be the only point through which three different diameters pass. The question is open whether there exist convex figures which possess a point through which exactly n diameters pass, if $n\geq 4$.

J. W. Andrushkiw (Newark, N.J.)

7056:

Kozinec, B. N. On the area of the kernel of an oval. Leningrad. Gos. Univ. Uč. Zap. Ser. Mat. Nauk 33 (1958), 83–89. (Russian)

Let K be a compact convex subset of the plane, with non-empty interior, and let $A(K)$ denote its area. For each point $p\in K$ let $K_p=K\cap(2p-K)$ be the maximal subset of K centered at p. It is well known that for each K the inequality $\max\{A(K_p);p\in K\}\geq 2A(K)/3$ holds [A. S.

Besicovitch, J. London Math. Soc. **23** (1948), 237–240; MR **10**, 320; in I. M. Yaglom and V. G. Boltyanskiĭ, *Vypuklye figury*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951 [MR **14**, 197] this result is attributed to S. S. Kovner; in the present paper it is stated that the result is to be found in M. A. Lavrent'ev and L. A. Lyusternik, *Osnovy variacionnogo isčisleniya*, vol. I, Moscow-Leningrad, 1935]. It was conjectured by E. Ehrhart [C. R. Acad. Sci. Paris **241** (1955), 274–276; MR **17**, 350] that if g denotes the centroid of K then $A(K_g) \geq 2A(K)/3$; Ehrhart also proved this inequality under some assumptions on the set K. In the present paper Ehrhart's conjecture is established without any restrictions. The same result was independently obtained by B. M. Stewart [Pacific J. Math. **8** (1958), 335–337; MR **20** #4238]. *B. Grünbaum* (Seattle, Wash.)

7057:

Mirsky, L. Problems of arithmetical geometry. Math. Gaz. **44** (1960), 182–191.

A lecture about some related problems. The problem to determine the network of minimum length linking n points within the unit square; the smallest circle covering a region of given diameter; the overlap problem; the turning problem of Kakeya. *O. Bottema* (Delft)

7058:

Melzak, Z. A. The isoperimetric problem of the convex hull of a closed space curve. Proc. Amer. Math. Soc. **11** (1960), 265–274.

Bonnesen posed the problem of finding a closed curve in E^3 with a given length for which the volume of its convex closure is maximal. The analogous problem in E^{2n} was solved by Schoenberg [Acta Math. **91** (1954), 143–164; MR **16**, 508] under certain restrictions. Also, the problem for open C in E^3 was solved by Egerváry [Publ. Math. Debrecen **1** (1949), 65–70; MR **12**, 46]. The original problem offers peculiar difficulties and is solved here under the following strong restrictions: C is of class C^1, has the planes $x=0$ and $y=0$ as planes of symmetry, its projections on these planes are open convex arcs, and its projection on $z=0$ is a closed convex curve. If C is given by $x(s)$, $y(s)$, $z(s)$, the volume of its convex closure is maximal if and only if it is similar to a periodic solution of the equations $x'=-xy^2$, $y'=-yx^2$, $z'=xy$. *H. Busemann* (Los Angeles, Calif.)

7059:

Fejes Tóth, L. An arrangement of two-dimensional cells. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 61–64.

A principal section of a natural honeycomb with a perceptible thickness of wax may be described as a "network" consisting of a connected region bounded by at least two closed curves. The "thickness" of the network is the minimal distance between points of two distinct bounding curves. The rest of the plane consists of a set of regions called the "cells" of the network. The author considers a network of thickness $2t$ contained in a convex hexagon of area H and having n convex cells of area at least h. He proves that

$$n^{1/2} \leq (H^{1/2}-12^{1/4}t)/(h^{1/2}+12^{1/4}t).$$

H. S. M. Coxeter (Toronto)

7060:

Fejes Tóth, L. Einige "schöne" Extremalfiguren. Arkhimedes **1959**, no. 2, 1–10. (Finnish)

The author summarizes a number of his earlier papers [Jber. Deutsch. Math.-Ver. **53** (1943), 66–68; MR **8**, 167; Acta Math. Acad. Sci. Hungar. **4** (1953), 103–114; **5** (1954), 41–44; **7** (1956), 95–98, 397–401; MR **15**, 341; **16**, 65; **18**, 63; **21** #5937; Publ. Math. Debrecen **5** (1957), 119–127; MR **19**, 763; Elemente der Math. **13** (1958), 32–34; MR **19**, 1074; Publ. Math. Debrecen **6** (1959), 234–240; MR **22** #4017]. Among the many beautiful figures are the square antiprism and snub cube whose vertices provide the "best" distribution of 8 or 24 points on a sphere, and the solutions of various problems of packing and covering, especially the densest packing and thinnest covering of the hyperbolic plane by horocycles. *H. S. M. Coxeter* (Toronto)

DIFFERENTIAL GEOMETRY

See also 6920, 6921, 7112, 7140, 7142, 7147, 7150.

7061:

Laugwitz, Detlef. ★Differentialgeometrie. Mathematische Leitfäden. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1960. 183 pp. DM 24.60.

This slim textbook contains an introduction to classical differential geometry, tensor algebra and analysis, Riemannian geometry and differential geometry in the large. The highlights of these subjects are developed in a careful, systematic and unified manner, first using vector techniques and gradually introducing tensor notation and methods in connection with classical surface theory. Most of the examples are really extensions of the theory. Unusual are the wide range of subjects treated and the discussion of some topics (analytical dynamics, metric spaces) rarely developed in an introductory text. An outline of the contents follows: (I) Local differential geometry of space curves: Differential geometric properties of curves. Complete system of invariants of a space curve. (II) Local differential geometry of surfaces: Surfaces and surface curves. Intrinsic geometry of surfaces. Curvature theory of surfaces. Special questions of surface theory. (III) Tensor analysis and Riemannian geometry: Differentiable manifolds. Tensor algebra. Tensor analysis. Geometry of affinely connected spaces. Foundations of Riemannian geometry. (IV) Further development and application of Riemannian geometry: Spaces of constant curvature and non-euclidean geometry. Mappings. Riemannian spaces in analytical dynamics. Metric differential geometry and the characterization of Riemannian geometry. (V) Topics in differential geometry in the large: Curves in the large. Surfaces in the large. *A. Fialkow* (Brooklyn, N.Y.)

7062:

Blaschke, Wilhelm; Reichardt, Hans. ★Einführung in die Differentialgeometrie. 2te Aufl. Die Grundlehren der mathematischen Wissenschaften, Bd. 58. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960. vii+173 pp. DM 24.00.

The first seven chapters reproduce the first edition of this book (1950) without essential change; see MR **13**, 274.

There are some references to the newer literature, and, in an appendix, thirty of the short notes of explanation prepared by A. P. Norden for the translation of the first edition into Russian.

The eighth chapter, written by Reichardt, deals in twenty pages with m-dimensional manifolds in n-dimensional Euclidean space. The point of view is, as in the first seven chapters, that of working independently of parameter representations. Some tensor calculus is assumed, but the extension of the Cartan calculus is presented. Tensor fields and their differentials are broken into tangential and normal components; a linear connection is presented; the Gauss-Codazzi equations are generalized; the higher osculating spaces are introduced and with them a finer classification of normal components.

A. Schwartz (New York)

7063:

Lagrange, René. Sur les sphères osculatrices et les arêtes de courbure successives d'une courbe. J. Math. Pures Appl. (9) **39** (1960), 33–62.

Let $C_0, C_1, \cdots$ denote curves in euclidean n-space E_n. Let C_{k+1} be the locus of the centers of the osculating hyperspheres of C_k, $k=0, 1, \cdots$. Thus C_0 and C_{2k} have the same moving n-hedra at corresponding points. The author studies the equation $C_0=C_{2k}$.

"Le premier chapitre est consacré à E_3 et le deuxième à E_n ($n>3$). On verra que, dans E_3, les courbes en question relèvent d'une équation différentielle linéaire du quatrième ordre dont la difficulté de résolution est pratiquement indépendante de l'ordre $2k$. Dans E_n on est conduit à une équation différentielle linéaire d'ordre $n-2$ et des équations du type $Vy=\alpha y$, où α est une racine k-ième de 1, et où V est le produit de deux opérateurs différentiels linéaires d'ordre $n-1$. L'inconnue est la première courbure, tandis que les opérateurs ne sont fonctions que des rapports des courbures. Ces résultats sont précisés pour certaines hélices de E_5."

P. Scherk (Toronto)

7064:

Parodi, Maurice. Sur la détermination de propriétés géométriques des courbes intégrales de certaines équations différentielles. Bull. Sci. Math. (2) **83** (1959), 123–128.

Dans un article antérieur [même Bull. (2) **82** (1958), 14–16; MR **21** #5208] l'auteur, s'appuyant sur un théorème de J. Hadamard donnant des conditions suffisantes de nullité pour un déterminant, a fait connaître une méthode générale permettant de déterminer les courbes planes définies par une inégalité entre les valeurs absolues de fonctions données de leurs éléments de contact. Il envisage ici le problème inverse, qui consiste à déduire de la proposition de Hadamard des propriétés des courbes intégrales de certaines équations différentielles. Les équations considérées sont des équations d'ordre p de la forme

$$(1)\quad x^n+\varphi_1(x, y, y', \cdots, y^{(p)})x^{n-1}+\cdots \\ +\varphi_n(x, y, y', \cdots, y^{(p)}) = 0,$$

où n est un entier positif et $\varphi_1, \varphi_2, \cdots, \varphi_n$ des fonctions données. En considérant (1) comme une équation algébrique où l'inconnue est x, et en mettant son primer membre sous forme de déterminant, on peut, en invoquant le théorème de Hadamard, affirmer que l'équation différentielle (1) ne peut être satisfaite si l'on a simultanément

$$|x| > 1, \quad |x+\varphi_1| > \sum_{k=2}^{n} |\varphi_k|.$$

La deuxième de ces inégalités est susceptible d'exprimer une certaine propriété géométrique P des courbes intégrales de (1). On peut dire alors que P ne peut appartenir aux courbes intégrales de (1) si $|x|>1$, et que si ces courbes possèdent P ce ne peut être qu'en dehors de l'intervalle $|x|>1$. Sur l'intervalle $|x|>1$ on a donc la propriété contraire à P. L'auteur illustre ces résultats par deux exemples, dont l'un, relatif au cas où les φ_i ne contiennent pas x, l'amène à envisager la circonstance où l'équation algébrique (1) serait abélienne. L'invariance de (1) pour certaines substitutions permet alors de définir pour chacune d'elles deux nouvelles propriétés géométriques P_1, P_2 des courbes intégrales de (1) ne pouvant pas se présenter simultanément.

P. Vincensini (Marseille)

7065:

Saban, Giacomo. Détermination des congruences synéctiques et des surfaces réglées. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **21** (1956), 195–199 (1957). (Turkish summary)

Bekanntlich läßt sich eine gesuchte Gerade im Euklidischen Raum durch einen dualen Einheitsvektor $\mathfrak{A}=\mathfrak{a}+\varepsilon\bar{\mathfrak{a}}$ mit $\varepsilon^2=0$ darstellen. Ist $\mathfrak{A}$ Funktion eines dualen Parameters, so entsteht ein synektisches Strahlensystem [s. etwa Blaschke, *Vorlesungen über Differentialgeometrie*, Bd. I, 3te Aufl., Dover, New York, 1945; MR **7**, 391]. Verf. zeigt: Ein solches System ist durch eine duale Invariante vollständig bestimmt. Das sphärische Bild, welches stets in eine Kurve entartet, bestimmt das synektische System bis auf einen Parameter. Die Schlußbemerkung des Verf., daß jede Regelfläche durch ihr sphärisches Bild bis auf einen Parameter bestimmt ist, kann Ref. nicht einsehen.

W. Haack (Zbl **79**, 379)

7066:

Lumiste, Yu. G. Differential geometry of line surfaces V_3 in R_4. Mat. Sb. (N.S.) **50 (92)** (1960), 203–220. (Russian)

On étudie la théorie des congruences de droites ∞^3R_1 en R_4 par le méthode de G. Lapteff: prolongements successifs du système et construction des objets géometriques au voisinage de chaque ordre. La correspondance entre les rayons de la congruence ($\mathbf{e}_0$) et les points $\mathbf{m}=\mathbf{e}_0$ de l'hypersphère unité S_3 définit l'image hypersphérique S_2 de V_3. La forme métrique et la seconde forme quadratique de S_2

$$d\varphi^2 = (d\mathbf{e}_0 d\mathbf{e}_0) = \gamma_{ij}\omega^i\omega^j, \quad \chi^{(2)} = (d^2\mathbf{e}_0\mathbf{n}) = \beta_{ij}\omega^i\omega^j$$

avec les

$$\psi^{(2)} = \mu_{ij}\omega^i\omega^j, \quad \theta^{(1)} = \nu_i\omega^i,$$

où $\nu_i=(\mathbf{e}_i\mathbf{n})$, composent l'ensemble des formes invariantes de V_3 qui définissent la congruence V_3 en R_4 à un déplacement près, si les équations de structure sont vérifiées. Pour le repère holonome $\omega^i=du^i$ elles réduisent à trois équations de Gauss, et Peterson-Codazzi pour $\hat{S}_2$, une équation de Sannia généralisée et l'équation $\varepsilon^{ij}\nu_{i;j}=$

$\varepsilon^{ij}\varepsilon^{kl}\mu_{kj}\beta_{li}$, ε^{ij}—tenseurdiscriminant. En étudiant le voisinage du rayon l'auteur généralise les notions habituelles de congruence; par exemple, la notion du paramètre de la distribution d'une surface réglée $V_2 \subset V_3$. Il existe ici des paramètres tangentiels et normals; les foyers du rayon et pseudo-foyers dont le déplacement infinitésimal ne quitte pas le plan du rayon et la normale de l'image hypersphérique, etc. *S. P. Finikov* (Moscow)

7067:

Geĭdel'man, R. M. Theory of analytic congruences of planes in complex and dual unitary non-euclidean spaces and the projective theory of congruences of pairs of planes. Mat. Sb. (N.S.) **49 (91)** (1959), 281–316. (Russian)

The first chapter deals with families of $(m-1)$-dimensional planes (l) depending on parameters u^κ, $\kappa=1, 2, \cdots, a$, and lying in ordinary projective space P_n. The moving reference system consists of $n+1$ points A_α, $\alpha=1, 2, \cdots, n+1$, of which the A_i, $i=1, \cdots, m$, lie in the plane, and the A_p, $p=m+1, \cdots, n+1$, outside of it. Then $dA_\alpha=\omega_\alpha{}^\beta A_\beta$, α, β, $\gamma=1, 2, \cdots, n+1$, with $D\omega_\alpha{}^\beta=[\omega_\alpha{}^\gamma\omega_\gamma{}^\beta]$. For the plane l we have $\omega_i{}^p=0$. We can write $\omega_i{}^p=\lambda^p_{i\kappa_1}du^{\kappa_1}$, $\kappa_1=1, \cdots, a$. Continuation of this system leads, with the aid of Cartan's lemma, to equations of the form $d\lambda^p_{i\kappa_1}-\lambda^p_{j\kappa_1}\omega_i{}^j+\lambda^q_{i\kappa_1}\omega_q{}^p=\lambda^p_{i\kappa_1\kappa_2}\,du^{\kappa_2}$. Continuing this procedure ν times, we obtain equations of the form

$$d\lambda^p_{i\kappa_1\cdots\kappa_\nu}-\cdots=\lambda^p_{i\kappa_1\kappa_2\cdots\kappa_{\nu+1}}\,du^{\kappa_{\nu+1}},$$

where the λ are symmetrical in all κ ($\kappa_1, \cdots, \kappa_{\nu+1}=1, \cdots, a$). The quantities $\lambda^p_{i\kappa_1}$, $\lambda^p_{i\kappa_1\kappa_2}$, $\cdots$, $\lambda^p_{i\kappa_1\cdots\kappa_\nu}$ are the components of the fundamental object of the planes (l) and when we give a field of these objects of some rank k the family of planes is determined but for a projective transformation of the P_n [G. F. Laptev, Trudy Moskov. Mat. Obšč. **2** (1953), 275–382; MR **15**, 254].

It is now possible to define focal families of rank r. When $r=m-2$, the highest rank, we have developable surfaces. When the number of planes through a point of the P_n is finite, the family is a congruence. When the number in a hyperplane of the P_n is finite we have a pseudocongruence. Focal congruences and pseudocongruences are investigated, as well as stratified pairs and families of pairs of planes, where a pair is a set composed of an $(m-1)$-dimensional plane l and an $(n-m)$-dimensional plane L.

In the second chapter an analogous theory is developed for analytic congruences and pseudocongruences in a dual unitary non-euclidean space $K_n(e)$ in the sense of B. A. Rozenfel'd [*Neevklidovy geometrii*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; MR **17**, 293]. These spaces are based on the algebra of numbers $x=a+be$, $e^2=1$, and a function $y=f(x)$ is analytic, if, in the units $e_1=\frac{1}{2}(1+e)$, $e_2=\frac{1}{2}(1-e)$, it satisfies $\partial y_1/\partial x_2=\partial y_2/\partial x_1=0$. In the last chapter the theory is extended to the complex dual unitary spaces ${}^1K_n(i)$ of Rozenfel'd.

D. J. Struik (Cambridge, Mass.)

7068:

Matsumura, Sōji. Differentialgeometrie der Kugelscharen. I. J. Osaka Inst. Sci. Tech. Part I **8** (1958), no. 2, 13–41.

L'auteur dans la première partie de son article nous initie à la géométrie différentielle des familles de cercles. Après la définition des coordonnées pentasphériques et des transformations de Moebius il fait connaître la méthode de la détermination des invariants des familles de cercles. Il introduit la notion de la dérivation modifiée, puis il déduit les paramètres invariants des familles. Il détermine les équations différentielles fondamentales des familles de cercles, puis il écrit leur forme spéciale dans le cas de l'emploi des sphères de Vessiot. Dans le paragraphe suivant nous trouvons les invariants des familles de cercles relatifs au paramètre général, puis une condition suffisante et nécessaire afin qu'une famille de cercles soit composée des cercles normaux d'une famille de sphères. Ensuite il s'occupe de l'étude des points focaux des familles de cercles.

Il rend difficile la lecture de l'article que dans le texte se trouvent beaucoup de fautes de types différents et quelques formules sont incomplètes. Les références bibliographiques sont un peu négligées. *J. Merza* (Debrecen)

7069:

Godeaux, Lucien. Sur les suites de Laplace et sur les congruences *W*. Arch. Math. **11** (1960), 72–76.

Author's summary: "Dans cette note, nous considérons une forme de la condition pour qu'un réseau soit conjugué à une congruence donnée dans un espace projectif quelconque. Nous démontrons ensuite qu'une congruence W, dont (x) est une surface focale, étant donnée, il existe une infinité de congruences W ayant également (x) comme surface focale et dont les complexes linéaires osculateurs sont en involution avec celui de la congruence donnée." *T. K. Pan* (Norman, Okla.)

7070:

Godeaux, Lucien. Sulle superficie associate ad una successione di Laplace chiusa. Boll. Un. Mat. Ital. (3) **15** (1960), 159–161. (French summary)

Author's summary: "On démontre qu'une surface associée à une suite de Laplace terminée est nappe focal de deux congruences W." *T. K. Pan* (Norman, Okla.)

7071:

Drăgilă, Pavel. Sur les transformations T des congruences de droites. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 1049–1062.

M. S. Finikoff a défini la transformation T des congruences de droites [Ann. Scuola Norm. Sup. Pisa (2) **2** (1933), 59–88], transformation qui a d'ailleurs fait l'objet de diverses études, notamment de la part de M. R. Calapso [Mat. Sb. **42** (1935), 451–458]. Dans son mémoire, M. Finikoff utilise une normalisation des paramètres à partir des asymptotiques d'une nappe focale d'une certaine congruence W, et à condition d'admettre que les paramètres puissent être imaginaires, les conclusions ne dépendent pas de la réalité des asymptotiques. L'auteur du présent article, estimant que M. Finikoff s'est borné au cas où les asymptotiques sont réelles, reprend l'étude lorsque ces lignes sont imaginaires et en déduit des formes particulières des résultats obtenus dans le cas général par M. Finikoff. *M. Decuyper* (Lille)

7072:

Drăgilă, Pavel. Sur un système de couples de réseaux parallèles. Proc. Amer. Math. Soc. **11** (1960), 255–264.

Dans cet article l'auteur complète une étude antérieure [mêmes Proc. **10** (1959), 366–368; MR **21** #7511], relative aux couples de surfaces $(S, \bar{S})$ en correspondance ponctuelle avec parallélisme des tangentes homologues issues de deux points correspondants. Dans une telle correspondance, deux réseaux conjugués à invariants égaux R, $\bar{R}$ de S et $\bar{S}$ se correspondent par parallélisme des tangents. L'auteur identifie les couples de surfaces $(S, \bar{S})$ donnant lieu à la circonstance indiquée avec des couples transformés de Christoffel étudiés à un autre point de vue par l'auteur de la présente analyse. Il en signale des propriétés intéressantes, telle par exemple la propriété suivant laquelle les droites joignant les couples de points correspondants M, $\bar{M}$ de S et $\bar{S}$ engendrent des congruences dont les segments focaux sont partagés harmoniquement par les points M et $\bar{M}$. La considération du cas où les congruences engendrées par les tangentes aux courbes du réseau R (dont le support S est supposé à courbure totale négative) sont des congruences W (R est alors un réseau de Tzitzeïca-Demoulin) lui fournit l'occasion de compléter sur certains points l'exposition de G. Tzitzeïca de la théorie des réseaux en question. Il montre, en particulier, qu'il y a lieu d'associer aux réseaux R portés par des surfaces à courbure totale négative et aux congruences W correspondantes, des réseaux $\bar{R}$ portés par des surfaces $\bar{S}$ à courbure totale positive et les congruences $\bar{W}$ formées par leurs tangentes, les réseaux R et $\bar{R}$ donnant lieu à un parallélisme remarquable de propriétés. *P. Vincensini* (Marseille)

7073:

Vrănceanu, G. Sur les correspondances entre deux plans projectifs. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2** (**50**) (1958), 225–236.

L'auteur a montré [Boll. Un. Mat. Ital. (3) **12** (1957), 145–153, 489–506; MR **19**, 764, 1075] qu'on peut associer à une correspondance entre deux espaces projectifs un tenseur $\Pi_{jk}{}^{i}$; maintenant il applique sa théorie au cas des plans. Les courbes caractéristiques d'une correspondance de la première espèce étant données, alors la correspondance est déterminée, abstraction faite de quatre constantes au plus. En étudiant le cas des courbes caractéristiques des droites on fait une étude complète du cas où ces droites passent par trois points situés en ligne droite. *A. Švec* (Prague)

7074:

Kallenberg, G. W. M. Ruled surfaces in a particular geometry. Nederl. Akad. Wetensch. Proc. Ser. A **63** = Indag. Math. **22** (1960), 291–296.

Dans un précédent mémoire [mêmes Proc. **60** (1957), 147–158; MR **19**, 450] l'auteur a défini la géométrie différentielle d'un espace à trois dimensions où l'absolu est constitué par un plan, une droite de ce plan et un point de cette droite. Une métrique simple a été introduite; en coordonnées non homogènes, la distance du couple ordonné de points (x_1, x_2, x_3), (y_1, y_2, y_3) est $y_3 - x_3$ si cette différence n'est pas nulle, l'angle des deux vecteurs (l_1, l_2, l_3), (m_1, m_2, m_3) est $m_2/m_3 - l_2/l_3$; une théorie des courbes et des surfaces a été construite. Le présent mémoire est consacré à l'étude de certaines propriétés des surfaces réglées dans cette géométrie; ainsi l'auteur définit la ligne de striction, le paramètre de distribution d'une surface réglée et ces notions présentent des propriétés voisines de celles des notions analogues de la géométrie euclidienne. On donne aussi la "fonction caractéristique" d'une surface réglée, suivant la notion de "fonction caractéristique" d'une surface quelconque introduite dans le précédent mémoire. *M. Decuyper* (Lille)

7075:

Nevanlinna, Rolf. On differentiable mappings. Analytic functions, pp. 3–9. Princeton Univ. Press, Princeton, N.J., 1960.

This paper presents a somewhat simplified version of the author's previously published proof of Liouville's theorem, which states that a conformal mapping of Euclidean space of dimension $n \geq 3$ onto itself is composed of a translation, a rotation, possibly an inversion, and a homothetic transformation [Rev. Fac. Sci. Univ. Istanbul. Sér. A **19** (1954), 133–139; MR **17**, 525]. The mapping is assumed to be four times differentiable; the space may be a Hilbert space ($n = \infty$). The technique is that of the coordinate-free "absolute analysis," as developed in the recent book of that name by the brothers Nevanlinna [Springer, Berlin, 1959]. *A. E. Taylor* (Los Angeles, Calif.)

7076:

Temple, G. ★Cartesian tensors: An introduction. Methuen's Monographs on Physical Subjects. Methuen & Co., Ltd., London; John Wiley & Sons, Inc., New York; 1960. vii + 92 pp. 12s. 6d.

After a preliminary discussion of the elementary ideas on vectors, bases and orthogonal transformations, the author introduces a tensor as a multilinear function of direction and outlines the elementary operations of combination and differentiation, indicating some of the applications to classical and fluid dynamics and elasticity. The book also includes chapters on isotropic tensors, spinors, and orthogonal curvilinear coordinates.

In the section on isotropic tensors the author performs a service in drawing attention to Weyl's method for finding polynomial invariants of a system of vectors under the orthogonal group. The isotropic tensors of higher order are derived by the method used by G. F. Smith and R. S. Rivlin [Quart. Appl. Math. **15** (1957), 308–314; MR **21** #689] for the anisotropic tensors and an explicit reference to their work would perhaps have been useful.

There are a number of misprints and in some places the text is a little misleading. For example, in the section on the strain tensor the impression is given that only the linear terms of classical elasticity are subject to compatibility conditions. Here, and in the chapter on isotropic tensors, there appears to be some confusion of notation. Despite these minor faults, the book is valuable in presenting an interesting approach to a number of aspects of the subject. *J. E. Adkins* (Providence, R.I.)

7077:

Aczél, J. Verallgemeinerte Addition von Dichten. Publ. Math. Debrecen **7** (1960), 10–15.

The author deals with the problem of when two G-objects $\mathfrak{g}_1, \mathfrak{g}_2$ (these are geometric objects with one component with the transformation formulae $\bar{\mathfrak{g}}_i=\theta_i[\theta_i^{-1}(\mathfrak{g}_i)J]$, where θ_i are invertible functions and J is the Jacobian of the coordinate transformation) admit a generalized addition, i.e., when

$$\mathfrak{g}_3 = F_3[F_1^{-1}(\mathfrak{g}_1)+F_2^{-1}(\mathfrak{g}_2)],$$

where F_i are invertible functions (in the paper regarded as given), is again a G-object. (For all F_i equal this problem has been solved under differentiability conditions by S. Gołąb and H. Pidek-Łopuszańska [Ann. Polon. Math. **4** (1958), 226–248; MR **20** #4282].) This problem leads to a functional equation, which is solved under the assumption of measurability of θ's and F's. The solution is: $\theta_i(x)=F_i(k_i|x|^a \operatorname{sg} x+K_i)$ $(i=1, 2; k_i\neq 0, a\neq 0)$, $\theta_3(x)=F_3(k_3|x|^a \operatorname{sg} x+K_1+K_2)$ $(k_3\neq 0)$; and if we confine ourselves to coordinate transformations with positive J, also $\theta_i(x)=F_i(k_i \log x+K_i)$ $(i=1, 2;\ k_i\neq 0,\ x>0)$, $\theta_3(x)=F_3[(k_1+k_2)\log x+K_3]$ $(k_1+k_2\neq 0)$.

M. Kuczma (Kraków)

7078:

Yano, K.; Davies, E. T. On some local properties of fibred spaces. Kōdai Math. Sem. Rep. **11** (1959), 158–177.

A differentiable manifold E of dimension $m+n$ is fibred by means of an equivalence relation R so that the fibres are m-dimensional submanifolds over the base space $X=E/R$. The tangent space to E at any point is decomposed (in a way not clear to the reviewer) into two complementary spaces, the vertical vectors tangential to the fibre and the horizontal vectors transversal to the fibre. Conditions are found for the distribution of horizontal vectors to be invariant for displacement along the fibre. Conditions in terms of Lie derivatives are obtained in order that a tensor field over E shall determine uniquely a tensor field over X, and similar conditions are obtained in order that an affine connexion over E shall determine an affine connexion over X. A similar problem is solved when E carries a system of paths, and also when E has a riemannian structure. In particular, conditions are obtained for the isometry of the fibres at two "infinitesimally near" points. This is followed by the treatment of a Finsler space as a fibred space where E, the tangent bundle of the base space X, carries a suitable metric and connexion.

Although the terminology of fibre spaces is used, the paper is concerned with only the local properties of such spaces and this justifies the implicit assumption that the fibred space is parallelisable. The authors use systematically the ideas and techniques of classical tensor calculus.

T. J. Willmore (Liverpool)

7079:

Stavroulakis, Nicias. Espaces logarithmiques à un nombre quelconque de dimensions. Points logarithmiques dans les espaces de Riemann à $n\geqq 3$ dimensions. Rend. Mat. e Appl. (5) **18** (1959), 283–312.

In his previous papers [C. R. Acad. Sci. Paris **246** (1958), 1149–1152, 1368–1371; MR **20** #2749a, b] the author studied properties of riemannian spaces of two dimensions which admit isolated singularities of a special kind. In the present paper he shows, by suitably generalizing the concepts of logarithmic sheet and logarithmic cylinder, that the results of the earlier papers can be extended to apply to riemannian spaces of dimension $\geqq 3$.

T. J. Willmore (Liverpool)

7080:

Vitner, Čestmír. Aussergewöhnliche Punkte auf Kurven in Riemannschen Räumen. Časopis Pěst. Mat. **84** (1959), 433–453. (Czech. Russian and German summaries)

In den aussergewöhnlichen Punkten einer Kurve im Riemannschen Raume V_n (in welchen die absoluten Ableitungen $M', \cdots, M^{(n)}$ linear abhängig sind) definiert man Krümmungen und Normalen als rechtsseitige Grenzwerte der gewöhnlichen Krümmungen und Normalen; man gibt ihre explicite Formeln und die verallgemeinerten Frenetsche Gleichungen.

A. Švec (Prague)

7081:

Moór, Arthur. Erweiterung des Begriffs der Räume skalarer und konstanter Krümmung. Acta Sci. Math. Szeged **21** (1960), 53–77.

A Riemannian manifold with positive definite line element is said to have scalar curvature if the curvature tensor is a scalar function times the proper combination of the metric tensor. By Schur's theorem, if the dimension is greater than two, such a space has constant curvature. The author gives four generalizations of this definition to positive definite Riemannian spaces with an affine connection, all of which reduce to constant curvature for the natural connection. Four distinct generalizations of Schur's theorem are proved. Some relations between the different definitions are discussed and examples given.

L. W. Green (Minneapolis, Minn.)

7082:

Boboc, N. À propos des périodes d'un vecteur harmonique défini sur une variété riemannienne R_n. Com. Acad. R. P. Romîne **9** (1959), 893–898. (Romanian. Russian and French summaries)

Let R_n be a riemannian variety of class C^r $(r\geqq 2)$, with a metric $ds^2=a_{ij}dx^idx^j$, and denote by v_i a harmonic vector field, so that $v_{,i}{}^i=0$, $v_{,j}{}^i=v_{,i}{}^j$; also, let D be an orientable, n-dimensional, compact subvariety of boundary ∂D. The period of v_i with respect to ∂D is $p(v_i, \partial D)=\int_{\partial D} dv^*$ and vanishes if R_n is compact. The author shows that this statement remains valid also if R_n is not compact, provided only that its ideal boundary is of measure zero with respect to the given metric. For $n=2$ this result contains as a particular case a theorem of R. Nevanlinna.

E. Grosswald (Philadelphia, Pa.)

7083:

Green, Leon W. A sphere characterization related to Blaschke's conjecture. Pacific J. Math. **10** (1960), 837–841.

Suppose that every solution of the differential equation $y''(r)+K(r)y(r)=0$ is odd-harmonic, periodic of period 2π, $y(r_0)=0$ implies $y'(r_0+\pi/2)=0$, and $K(r)$ is continuous and of period π. Then it is proved that $K(r)$ is identically 1. Let S be a C^3-surface with a complete positive-definite Riemannian metric. Suppose that $R=R(s)$ is a geodesic ray at a point P with direction θ and arc-length s, e the initial element (P, θ) and $h=(P, \theta\pm\pi/2)$ an orthogonal element at P. $F(e)$ is called the focal distance for e, if

$R(F(e))$ is the first focal point along R to the element h. From the above result it follows that if $F(e)$ is finite and independent of the initial element e in the unit tangent bundle of S, then S has constant positive curvature. This theorem is a special case of the long-standing conjecture of Blaschke, in which the conjugate distance is used for $F(e)$. *C. C. Hsiung* (Bethlehem, Pa.)

7084:

Chern, S. S.; Hano, J.; Hsiung, C. C. A uniqueness theorem on closed convex hypersurfaces in Euclidean space. J. Math. Mech. **9** (1960), 85–88.

The following theorem is proved. Let Σ, Σ' be two closed, strictly convex, C^2 hypersurfaces in E^{n+1} $(n+1 \geq 3)$. Let $f:\Sigma \rightarrow \Sigma'$ be a diffeomorphism such that Σ and Σ' have parallel outward normals at $p \in \Sigma$ and $p'=f(p)$ respectively. Let $P_l(p)$ be the lth elementary symmetric function of the principal radii of curvature of Σ at p, and $P_l(p')$ the corresponding expression for Σ'. If for a fixed l, $2 \leq l \leq n$, we have $P_{l-1}(p) \leq P_{l-1}(p')$ and $P_l(p) \geq P_l(p')$ for all $p \in \Sigma$, then f is a translation. *C. B. Allendoerfer* (Seattle, Wash.)

7085:

Stong, Robert E. Some global properties of hypersurfaces. Proc. Amer. Math. Soc. **11** (1960), 126–131.

The theorems in this paper are generalisations from $n=3$ to general n of results of Voss [Math. Ann. **131** (1956), 180–218; MR **18**, 229], Hopf and Voss [Arch. Math. **3** (1952), 187–192; MR **14**, 583], and Hsü [same Proc. **10** (1959), 324–328; MR **21** #6601]. They are essentially contained in A. Aeppli, Comment. Math. Helv. **33** (1959), 174–195 [MR **22** #5945] which the author seems to have overlooked. Actually Aeppli goes farther inasmuch as he gives also uniqueness theorems in terms of the higher order symmetric functions of the principal curvatures, instead of the mean curvature only. *H. Busemann* (Los Angeles, Calif.)

7086:

Aleksandrov, A. D. Uniqueness theorems for surfaces in the large. VII. Vestnik Leningrad. Univ. **15** (1960), no. 7, 5–13. (Russian. English summary)

[Part VI: same Vestnik **14** (1959), no. 1, 5–13; MR **21** #5227.] Let $H(x, y, z)$ be defined for all (x, y, z) except $(0, 0, 0)$ and positive homogeneous of degree 1. Let either H possess a second differential d^2H, or let H be of class C^1 with almost everywhere generalized square integrable second derivatives (because of the square integrability the latter condition does not include the former) and form d^2H with the generalized derivatives. In either case one of the eigenvalues of d^2H vanishes because of the homogeneity. Assume that at each point, with the exception of a set of measure 0, either $d^2H \equiv 0$ or d^2H has two non-vanishing eigenvalues R_1, R_2 of opposite sign and that (uniformly) $A > |R_1R_2^{-1}| > A^{-1}$. Then H is linear.

The supporting functions H', H'' of two convex bodies K', K'' possess almost everywhere second differentials. Applying the theorem to the difference $H'-H''$ and to $\Delta R_1 = R_1' - R_1''$, $\Delta R_2 = R_2' - R_2''$ gives: if $H'-H''$ satisfies either of the two conditions, and if a.e. either $\Delta R_1 = \Delta R_2 = 0$ or $A > |\Delta R_1/\Delta R_2| > A^{-1}$ and $\Delta R_1 \cdot \Delta R_2 < 0$, then K'' originates from K' by a translation. Under somewhat stronger conditions the latter theorem is contained in the earlier parts of this series. *H. Busemann* (Los Angeles, Calif.)

7087:

Sacksteder, Richard. On hypersurfaces with no negative sectional curvatures. Amer. J. Math. **82** (1960), 609–630.

In this paper the following theorem is proved. Let M be a complete n-dimensional Riemannian manifold $(n \geq 2)$, $X: M \rightarrow E^{n+1}$ an isometric imbedding of class C^{n+1} of M into Euclidean space E^{n+1}. Then, if at each point of M the curvature in every direction is non-negative and not identically zero, $X(M)$ is a convex surface, that is, is the boundary of a convex body. This theorem in the case $n=2$ was proved by various authors (Hadamard, Stoker, Chern and Lashof). The case $n>2$ was considered by Van Heijenoort, but in place of the condition concerning curvature of the manifold M there was required the local convexity of $X(M)$.

The paper contains an appendix in which an example is constructed of a twice differentiable surface Φ with an elliptical point P_0, at which are attained the maximum mean and minimum Gaussian curvatures, but where no neighborhood of P_0 is a spherical surface. This example is of interest in connection with the following theorem of Weyl: If at a point P of a four-times continuously differentiable surface with positive curvature the maximum mean and minimum Gaussian curvatures are attained, then in a neighborhood of this point the surface is a sphere. *A. V. Pogorelov* (Kharkov)

7088:

Heinz, Erhard. Über die Differentialungleichung $0 < \alpha \leq rt - s^2 \leq \beta < \infty$. Math. Z. **72** (1959/60), 107–126.

In this article the familiar theorem of A. D. Aleksandrov is strengthened, according to which a convex surface with Gaussian curvature confined between positive bounds is smooth and strictly convex. (Nothing is assumed beforehand concerning the surface other than convexity.) The author proves the following theorem.

Let $z(x, y)$ be a twice differentiable function in the region Ω satisfying the inequalities

$$0 < \alpha \leq rt - s^2 \leq \beta < \infty, \quad |z(x, y)| \leq \delta < \infty$$

$(p = z_x,\ q = z_y,\ \cdots,\ t = z_{yy})$; let Ω_d be the subset of Ω consisting of the points distant from the boundary of Ω by no less than d. Then for any two points (x_0, y_0) and (x, y) of Ω_d one has the inequality

$$\begin{aligned}\Gamma_2(\alpha, \beta, \delta, d)\{\lambda_d([(x-x_0)^2+(y-y_0)^2]^{1/2})\}^{\nu/2} &\leq \{(p(x, y)-p(x_0, y_0))^2+(q(x, y)-q(x_0, y_0))^2\}^{1/2} \\ &\leq \Gamma_1(\alpha, \beta, \delta, d)\{(x-x_0)^2+(y-y_0)^2\}^{\mu/2},\end{aligned}$$

where $\mu = \sqrt{(\alpha/\beta)}$, $\nu = \sqrt{(\beta/\alpha)}$, $\Gamma_1(\alpha, \beta, \delta, d)$ and $\Gamma_2(\alpha, \beta, \delta, d)$ are continuous positive functions of the indicated arguments, and $\lambda_d(\rho) = \rho$ for $0 \leq \rho \leq \frac{1}{2}d$, $\frac{1}{2}d$ for $\frac{1}{2}d \leq \rho < \infty$.

The proof is based on a study of the Monge-Ampère equation

$$rt - s^2 = E(x, y) \quad (0 < \alpha \leq E(x, y) \leq \beta < \infty)$$

by passage to the conjugate-isothermic parameters u, v in accordance with the equality

$$r\,dx^2 + 2s\,dxdy + t\,dy^2 = \lambda(du^2 + dv^2)$$

and a subsequent examination of the equivalent elliptic system for the functions $x(u, v)$, $y(u, v)$, $p(u, v)$, $q(u, v)$.
A. V. Pogorelov (Kharkov)

7089:
Heinz, Erhard. Neue a-priori-Abschätzungen für den Ortsvektor einer Fläche positiver Gaussscher Krümmung durch ihr Linienelement. Math. Z. **74** (1960), 129–157.

All the known solutions of the problem of isometric imbedding of a two-dimensional Riemannian manifold M with positive curvature as a regular surface Φ in three-dimensional Euclidean space are connected with establishing a priori estimates for the second derivatives of the vector $r(u, v)$ of a point of the surface Φ depending on its line element ds^2. Such bounds have been obtained for a closed surface by H. Weyl, for a surface with boundary by the reviewer. In both cases the bounds depend on the derivatives of the coefficients E, F, G of the line element ds^2 of the surface Φ up to the fourth order. In the paper under review the bounds of the second derivatives of r are obtained depending only on the third derivatives of E, F, G. Namely, the following theorem is proved.

Let $r(u, v)$ be a twice-differentiable vector-function defined in a region Ω of the uv-plane, $ds^2=Edu^2+2Fdudv+Gdv^2$ the line element of the surface $\Phi: r=r(u, v)$. Suppose the coefficients of the form ds^2 are bounded, together with their derivatives up to the third order and the reciprocal of the discriminant of the form, by a constant a; the Gaussian curvature of the surface Φ is everywhere greater than $b>0$; and the integral mean curvature is bounded by a constant $c<\infty$. Then on the set of interior points of Ω at a distance no less than $\rho>0$ from its boundary an estimate can be made of the moduli of the second derivatives of $r(u, v)$ which depends only on the numbers a, b, c, ρ. Furthermore, for any positive number $\nu<1$ an a priori estimate can be made for the Hölder constants of the second derivatives of r with respect to the exponent ν which depends on the same constants and on the number ν.

By means of this theorem there is obtained in the paper a solution of the well-known problem of H. Weyl in the following form. A closed two-dimensional Riemannian manifold of class C^3, homeomorphic to the sphere and with positive Gaussian curvature, is isometrically imbeddable in three-dimensional Euclidean space as a closed convex surface of class $C^{2+\nu}$ $(\nu<1)$.
A. V. Pogorelov (Kharkov)

7090:
Borisov, Yu. F. On the connection between the spatial form of smooth surfaces and their intrinsic geometry. Vestnik Leningrad. Univ. **14** (1959), no. 13, 20–26. (Russian. English summary)

In this paper surfaces are considered of class $C^{1+\alpha}$, $\alpha>\frac{2}{3}$. These are surfaces which under suitable choice of x, y, z coordinate axes admit a local representation by an equation $z=z(x, y)$, where $z(x, y)$ is a function with continuous first derivatives which satisfy a Hölder condition with exponent α. For such surfaces a number of theorems on their extrinsic form are proved under conditions pertaining to their intrinsic metric. (1) If the surface F of class $C^{1+\alpha}$, $\alpha>\frac{2}{3}$, is a manifold of bounded intrinsic curvature, and the curvature of every region on the surface is positive [negative], then F is also a manifold of bounded extrinsic curvature. (2) If the surface F of class $C^{1+\alpha}$, $\alpha>\frac{2}{3}$, has a k-times differentiable metric $(k\geqq 5)$ and positive Gaussian curvature, then the surface is at least $(k-1)$-times differentiable. If the metric is analytic, the surface is analytic. (3) A closed convex surface of class $C^{1+\alpha}$, $\alpha>\frac{2}{3}$, is rigid.

The paper is closely related to previous work of the author [same Vestnik **13** (1958), no. 7, 160–171; no. 19, 45–54; **14** (1959), no. 1, 34–50; no. 13, 83–92; MR **21** #3032–3034; **22** #2956]. The basic tool is the parallel translation introduced by the author on $C^{1+\alpha}$ surfaces $(\alpha>\frac{2}{3})$.

{Reviewer's remark: Errors which have turned up in the previous articles may invalidate the statement of the results in the present paper.} *A. V. Pogorelov* (Kharkov)

7091:
Rybnikov, A. K. Immersion of a 3-dimensional space with affine connection with torsion in 7-dimensional affine space. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 205–218. (Russian)

O. Galvani a montré [J. Math. Pures Appl. (9) **25** (1946), 209–239; MR **9**, 380] qu'on peut réaliser un espace à connexion affine avec torsion A_n comme une certaine variété dans l'espace affine à n^2 dimensions. L'A. a trouvé que pour $n=3$ il existe une réalisation (qui dépend de 4 fonctions de 3 variables) déjà dans l'espace à 7 dimensions.
A. Švec (Prague)

7092:
Alexandrov, A. D. Modern development of surface theory. Proc. Internat. Congress Math. 1958, pp. 3–18. Cambridge Univ. Press, New York, 1960.

This is a very readable survey of the theory of general surfaces as developed by the author, Pogorelov and others. Since there are no proofs and the definitions are often descriptive rather than precise, the usefulness of the survey is greatly marred by the complete absence of references.

The topics are: (two-dimensional) manifolds of bounded intrinsic curvature, where—with the upper angle—the sum of the absolute values of the excesses is bounded for any set of non-overlapping geodesic triangles lying in a compact set; the approximation of such manifolds with polyhedral ones; manifolds with bounded extrinsic curvature, where the total area of the spherical image is finite, all areas being counted positive and as often as they occur.

Most novel are the indications regarding a theory of Borisov based on parallel displacement. For a surface of class C' assume that the square of the angle between two surface normals divided by the (intrinsic) distance of the corresponding points is bounded uniformly in every compact set. Then parallel displacement of a vector can be defined extrinsically and intrinsically, the latter by taking shortest connection as autoparallel and for other curves by approximation with geodesic polygons. The two definitions coincide. Although these surfaces in general do not have bounded curvature, so that the excess function cannot be extended to a totally additive set function, an analogue to the Gauss-Bonnet theorem holds, relating the total rotation of a vector under parallel displacement along a Jordan curve bounding a disc to an integral over the boundary of its spherical image.
H. Busemann (Los Angeles, Calif.)

7093:

Gołąb, S. Sur quelques propriétés des lignes géodésiques. Ann. Polon. Math. 8 (1960), 91–103.

In Acta Math. **66** (1935), 1–47, Busemann and Feller proved: On a convex surface C a shortest curve G issuing from (or passing through) a point p where the surface has a tangent plane T, has at p a tangent t and an osculating plane normal to T. The upper and lower curvatures of G at p are the same as those of the normal section of C through t. If the latter are finite, then the geodesic curvature of G at p, i.e., the curvature of the projection of G on T, is zero. They state erroneously (without indication of proof) that this remains true when the upper normal curvature is not finite. The author gives examples of a convex cylinder and of strictly convex surfaces, where the normal curvature in the direction of a tangent t to a geodesic G at a point p exists and is infinite and the geodesic curvature of G at p is infinite or finite and positive.

H. Busemann (Los Angeles, Calif.)

7094:

Künneth, Hermann. Dualisierbare Kurven im R_n. Math. Ann. **140** (1960), 198–226.

The author extends his earlier work on curves in 3-space [J. Reine Angew. Math. **201** (1959), 87–99; MR **21** #3868] to curves in the n-dimensional projective space R_n. A curve is considered as the locus of its k-dimensional osculating planes for $0 \leq k \leq n-1$, giving n loci $S_k(t)$, $S_i(t) \subset S_k(t)$ for $i<k$, where $S_0(t)$ is the point locus. For convenience put $S_{-1}(t)=\emptyset$ and $S_n(t)=R_n$. The behaviour of the $S_k(t)$ is subject to two involved conditions defining dualizable curves, whose principal implications are

$$\lim_{t_l \to t} S_k(t) \vee S_l(t_l) = S_{k+l+1}(t) \quad (k+l \leq n-1),$$

$$\lim_{t_l \to t} S_k(t) \wedge S_l(t_l) = S_{k+l-n}(t) \quad (k+l \geq n-1),$$

where $\wedge$ and $\vee$ mean the lattice-theoretical intersection and sum. The intersections of the $S_k(t)$ $(n-r+1 \leq k \leq n-1)$ with an r-flat E_r in R_n not intersecting any $S_{n-r}(t)$ form a dualizable curve in E_r, the trace of the original curve in E_r. Defining a cusp or a 0-cusp in an obvious way, the curve has at t_0 a k-cusp if its trace in some E_{n-k} has a 0-cusp. t_0 counts as a cusp of multiplicity m if there are m values of k for which it is a k-cusp. The k-cusps are the elementary singularities; the other singularities are accumulation points of cusps.

An arc without k-cusps $(0 \leq k \leq n-1)$ is normal and has order n. A dualizable curve is the closure of the union of a countable number of normal arcs. The nature of the elementary singularities is discussed; in particular, relations of the order to the multiplicities of the cusps. For example, at an elementary singularity with a cusp of multiplicity m, the curve has order at most $n+m$.

There are various other results; in particular, concerning the dual to a trace, the "dualizable curve" obtained by projecting the curve from E_r (with the previous notation). Under an elliptic correlation of R_n an r-flat E_r goes into an $(n-r-1)$-flat E_{n-r-1} with $E_r \cap E_{n-r-1}=\emptyset$, and this leads to the study of manifolds formed by r-flats (analogues to ruled surfaces).

H. Busemann (Los Angeles, Calif.)

7095:

Iseki, Kanesiroo. On certain properties of parametric curves. J. Math. Soc. Japan **12** (1960), 129–173.

The purpose of this paper is to study curvature for curves in euclidean space R^m which do not satisfy the classical regularity assumptions. For any continuous curve φ, a mapping from a closed interval I into R^m, define $\Theta(\varphi)$ as the lower limit of the integrated curvature of regular C^2 curves tending uniformly to φ. Let $\Omega(\varphi)$, the bend of φ, be the supremum of the sum of the exterior angles at the vertices of polygons inscribed in φ. Then $\Theta(\varphi)=\Omega(\varphi)$. The main theorem of the paper states that, if φ is light and continuous on the open interval K, then the bend of φ equals the length of γ, where $\gamma(t)$ is any curve (not necessarily continuous) on the unit $(m-1)$-sphere such that, for every t, $\gamma(t)$ is the limit of a sequence of unit vectors of the form

$$P_n = \frac{\varphi(t_n)-\varphi(t)}{|\varphi(t_n)-\varphi(t)|}, \quad t_n > t_{n+1} > t, \ \lim t_n = t.$$

W. H. Fleming (Providence, R.I.)

7096:

Nasu, Yasuo. On similarities and transitive abelian groups of motions in Finsler spaces. Kumamoto J. Sci. Ser. A 4, 103–110 (1959).

Results obtained by the reviewer for G-spaces [(1) *The geometry of geodesics*, Academic Press, New York, 1955; (2) Tôhoku Math. J. (2) **9** (1957), 56–67; MR **17**, 779; **20** #2772] are proved for complete Finsler spaces S, where the metric is given by a function $F(x, \xi)>0$ for $\xi \neq 0$ with $F(x, k\xi)=kF(x, \xi)$ for $k>0$. If the distance pq in S is defined as the greatest lower bound of the F-lengths of all curves from p to q, then the resulting space is isometric to the space derived from $\bar{F}(x, \xi)$, where $\bar{F}(x, \xi) \leq 1$ is the convex closure of $F(x, \xi) \leq 1$ in ξ-space [(1), p. 83]. This fact allows reduction of most assertions to the case of convex $F(x, \xi) \leq 1$.

A proper (local) similarity of S is a mapping $p \rightarrow p'$ of S on itself satisfying (locally) $p'q'=kpq$, $k \neq 1$. If S possesses a proper local similarity then it is locally Minkowskian if $k>1$, Minkowskian if $k<1$, and the similarity is global.

If S possesses a transitive abelian group of motions Γ and if for all $x \in S$ and $\phi \in \Gamma$ the relation $xx^\phi + x^\phi x^{\phi^2} = xx^{\phi^2}$ holds, then S is Minkowskian. The author also proves that the latter condition may be omitted if the ϕ are mappings of class C^1, but this follows at once from Lie group theory.

H. Busemann (Los Angeles, Calif.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 6738, 7132.

7097:

Haupt, Otto. Zur Verallgemeinerung des Zweischeitelsatzes bei ebenen Kurven. Arch. Math. **11** (1960), 294–297.

Consider a domain G in the plane which is bounded by a closed Jordan curve J; and a closed Jordan curve C in G. Let Γ be a class of curves $K \subset G \cup J$ with the following properties: K is either a closed Jordan curve with at most one common point with J, or K is a Jordan arc with endpoints on J and no other common points with J. Two distinct curves in Γ have at most two common points. If three distinct points x_i $(i=1, 2, 3)$ on C and sufficiently small neighborhoods U_i of x_i are given, then for any

$y_i \in U_i$ a curve in Γ through y_1, y_2 and y_3 exists. Finally, if y_i and y_i' are both close to x_i, then the curves in Γ through the y_i and the y_i' respectively are close together.

A Γ-vertex of C is a point x of C such that any neighborhood of x contains four points of C lying on the same curve in Γ. The principal part of the paper is a detailed sketch of a proof for the theorem that C has at least two Γ-vertices. *H. Busemann* (Los Angeles, Calif.)

7098:

Frolík, Zdeněk. Concerning topological convergence of sets. Czechoslovak Math. J. **10** (85) (1960), 168–180. (Russian summary)

The paper begins with a discussion of topological lim sup, lim inf and convergence of nets of subsets of a topological space. Besides the usual properties, it is shown that every net of subsets of a regular space has a convergent subnet. In a natural way, the classical notion of continuum of convergence is generalized by replacing sequences of continua by nets of continua. The following result is proved. Let K be a continuum (metrizability not assumed), and let N be the set of all points at which K is not locally connected. Then N is a union of continua of convergence in K. If R is the equivalence relation in K such that the equivalence classes are the components of $\bar{N}$ and the one-point subsets of $K-\bar{N}$, then the quotient space K/R is locally connected. This generalizes a classical theorem of R. L. Moore for metrizable K [cf. C. Kuratowski, *Topologie. II*, Monografie Matematyczne, Vol. 21, Warsaw-Wrocław, 1950; MR **12**, 517; pp. 176–178].

Ky Fan (Detroit, Mich.)

7099:

Frolík, Zdeněk. The topological product of two pseudocompact spaces. Czechoslovak Math. J. **10** (85) (1960), 339–349. (Russian summary)

The main result (in the reviewer's opinion) is a direct and somewhat shorter proof of Glicksberg's theorem [Trans. Amer. Math. Soc. **90** (1959), 369–382; MR **21** #4405] that $\beta(X\times Y)=\beta X\times\beta Y$ if and only if $X\times Y$ is pseudocompact. Three other equivalent conditions are given. Further, the author studies the class $\mathfrak{P}$ of spaces X such that $X\times Y$ is pseudocompact whenever Y is, and the class $\mathfrak{P}_F$ of spaces all of whose closed subspaces belong to $\mathfrak{P}$. For membership in $\mathfrak{P}_F$ the condition is just that every infinite subset has an infinite subset with compact closure. The analogous characterization of $\mathfrak{P}$ is much more complicated. For pseudocompact spaces, belonging to $\mathfrak{P}$ (hence also to $\mathfrak{P}_F$) is a local property. $\mathfrak{P}_F$ is closed under countable products, but not under arbitrary products. *J. R. Isbell* (Seattle, Wash.)

7100:

Frolík, Zdeněk. Generalizations of the G_δ-property of complete metric spaces. Czechoslovak Math. J. **10** (85) (1960), 359–379. (Russian summary)

The paper concerns absolute G_δ spaces and the generalization to higher cardinals, $G(\mathfrak{m})$ spaces. The main theorem characterizes completely regular absolute G_δ spaces, by the existence of a sequence of open coverings such that every open filter meeting all of these coverings has a cluster point. This generalizes directly to $G(\mathfrak{m})$ spaces. Lavrent'ev's theorem [C. R. Acad. Sci. Paris **178** (1924), 187–190] on mappings into metrizable absolute G_δ spaces generalizes also, with G_δ replaced by $G(\mathfrak{m})$ and the metric replaced by a "basic" family of $\mathfrak{m}$ open coverings. There are given a number of other results; for example, if a completely regular space has a sequence of collections of open subsets satisfying the conditions of the main theorem except that they cover only certain dense subspaces, then there is a dense absolute G_δ subspace.

J. R. Isbell (Seattle, Wash.)

7101:

Mardešić, Sibe. On covering dimension and inverse limits of compact spaces. Illinois J. Math. **4** (1960), 278–291.

Expansions of Hausdorff compact spaces into inverse sequences are considered. Dimension is taken in the covering sense. Theorem I: A compact space X can be expanded into a sequence of metrizable compact spaces of dimensions $\leqq \dim X$, the cardinality of the sequence not surpassing the weight of X (=minimal cardinality of an open basis). Theorem II: A mapping of a compact X into compact P can be factorized by means of an intermediate compact Q, such that $\dim Q\leqq\dim X$, weight $Q\leqq$ weight P, and such that the mapping used of X into Q is onto. Theorem I combined with one of H. Freudenthal [Compositio Math. **4** (1937), 145–234] yields an expansion of a compact X into a double sequence of polyhedra of dimensions not surpassing $\dim X$. From Theorem II a compactification X' of a normal X with $\dim X'\leqq\dim X$ and weight $X'\leqq$ weight X is derived.

Theorem III asserts for nonmetrizable compact Hausdorff X the existence of an expansion embracing compact spaces of dimensions $\leqq \dim X$ and of weight $<$ weight X. The conjecture that the same could be done using compact polyhedra is disproved by a counter-example of dimension 1 (and inductive dimension >1).

H. Freudenthal (New Haven, Conn.)

7102:

Kodama, Yukihiro. On a problem of Alexandroff concerning the dimension of product spaces. II. J. Math. Soc. Japan **11** (1959), 94–111.

[For part I see same J. **10** (1958), 380–404; MR **21** #5198.] A space X is "dimensionally full-valued" for a class P of spaces if, for each $Y\in P$, $\dim(X\times Y)=\dim X+\dim Y$, dim denoting the covering dimension. In part I the author had shown that, for an n-dimensional compact metric X to be dimensionally full-valued for the class of compact metric spaces, it is necessary and sufficient that, for each coefficient group $Z(\alpha)$ of a certain type, there exists a closed $A\subset X$ such that the Čech homology group $H_n(X, A; Z(\alpha))$ is non-trivial. Here he shows that this continues to hold if "compact metric" is replaced throughout by "locally compact fully normal", provided the Čech groups are taken "unrestricted" (based on all open coverings). The proof of sufficiency is extended to give the theorem [K. Morita, Amer. J. Math. **75** (1953), 205–223; MR **14**, 893] that every 1-dimensional fully normal space is dimensionally full-valued for the class Q of locally compact fully normal spaces. As a corollary of the main theorem it is shown that if X is n-dimensional and fully normal and contains a closed subset A such that $H_n(X, A; Z)\neq 0$ (Z=integers), then X is dimensionally full-valued for Q. This corollary is shown to include several significant special cases (e.g., X any CW complex),

but its converse is false. The author also gives some variations of his general criterion, and shows by examples that it cannot be significantly simplified. It is an open question whether Boltyanskiĭ's criterion, known to be equivalent to the author's in the compact metric case (see part I), is valid at the present generality.
A. H. Stone (Manchester)

7103:

Stone, A. H. Universal spaces for some metrizable uniformities. Quart. J. Math. Oxford Ser. (2) **11** (1960), 105–115.

The paper concerns existence of universal spaces in the class of all metric uniform spaces of density character m and satisfying various side conditions. Such spaces are found for the side conditions (a) the uniformity is the finest compatible with the topology, (b) every uniform covering has a Euclidean uniform refinement, (c) every uniform covering has a star-countable uniform refinement; also for (d) every uniform covering has a star-bounded uniform refinement, in case $m \geqq c$. Non-existence of a universal space is proved for $m=\aleph_0$ and (e) every uniform covering has a star-finite uniform refinement. The question is open for (d) and (e) with other cardinals m.
J. R. Isbell (Seattle, Wash.)

7104:

Treybig, L. B. Concerning certain locally peripherally separable spaces. Pacific J. Math. **10** (1960), 697–704.

F. B. Jones proved that every connected, locally connected, locally peripherally separable (l.p.s.) metric space is separable [Bull. Amer. Math. Soc. **41** (1935), 437–439]. He asked whether local connectedness can be omitted from the assumptions in his theorem [J. Elisha Mitchell Sci. Soc. **70** (1954), 30–33; MR **16**, 59]. The author answers the question in the negative by a rather involved example of a non-separable metric space X, which is connected and l.p.s.; moreover, X is semi-locally connected and fails to be locally separable in a set of points which is itself only separable. Furthermore, it is shown that a metric space which is compactly connected and l.p.s. is necessarily separable.
S. Mardešić (Zagreb)

7105:

Levine, Norman. Remarks on uniform continuity in metric spaces. Amer. Math. Monthly **67** (1960), 562–563.

Given a finite family of functions f_i continuous from the metric space S_1, with metric d_1, into the metric space S_2, with metric d_2, it is proved that d_1 may be replaced by an equivalent metric d_1^* such that the f_i are uniformly continuous from S_1 with respect to d_1^* and d_2; $d_1^*(a, b)$ is $d_1(a, b)+\sum_i d_2(f_i(a), f_i(b))$. An enumerable sequence of continuous functions f_i from S_1 to S_2 can all be made uniformly continuous, if both the metrics d_1, d_2 are replaced by equivalent ones; d_2 is first replaced by an equivalent bounded metric d_2^*, and then $d_1^*(a, b)$ is $d_1(a, b)+\sum_i 2^{-i}d_2^*(f_i(a), f_i(b))$.
V. S. Krishnan (Madras)

7106:

Müller, Günter. Allgemeine Konvergenzbegriffe in topologischen Vereinen und Verbänden. Math. Ann. **139**, 76–86 (1959).

The author's object is to include in a common generalization some general theories of convergence, due to Abian [Math. Ann. **134** (1957), 93–94; MR **20** #271], Dörge and Wagner [Math. Ann. **123** (1951) 1–33; MR **13**, 635], and Nöbeling. Given subsets $\mathfrak{S}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$, with $\mathfrak{B}_1 \subset \mathfrak{B}_2$, of a topological Boolean algebra $\mathfrak{B}$ [in the sense of Nöbeling, *Grundlagen der analytischen Topologie*, Springer, Berlin, 1954; MR **16**, 844], two definitions (inequivalent in general) are given of subsets $\mathfrak{H}$ and $\mathfrak{G}$ of $\mathfrak{B}$. In the applications, $\mathfrak{B}$ is a Hausdorff space, $\mathfrak{S}$ is a basis of open sets, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ roughly correspond to "almost all" and "cofinally often" in a directed set, and $\mathfrak{H}$, $\mathfrak{G}$ are roughly the upper and lower limit sets. The author derives some relations between these sets in the general setting, and gives conditions under which they reduce to atoms. Special cases of the theory include, beside the usual notion of convergence, the following notions: point of accumulation, tangent (or circle of curvature, etc.) to a curve at a point, and continuity.
A. H. Stone (Manchester)

7107:

Mrówka, S. Axiomatic characterization of the family of all clusters in a proximity space. Fund. Math. **49** (1959/60), 123–126.

A cluster [S. Leader, Fund. Math. **47** (1959), 205–213; MR **22** #2978] in a proximity space X is a class $\mathfrak{c}$ of subsets of X such that: (a) $A, B \in \mathfrak{c}$ implies A is close to B. (b) A close to each B in $\mathfrak{c}$ implies $A \in \mathfrak{c}$. (c) $A \cup B \in \mathfrak{c}$ implies $A \in \mathfrak{c}$ or $B \in \mathfrak{c}$. Theorem: If X is a proximity space and $\mathfrak{C}$ the family of all clusters in X, then $\mathfrak{C}$ is a family of semi-ultrafilters of X satisfying (a), (b), (c). Theorem: If $\mathfrak{C}$ is a family of semi-ultrafilters on a set X, satisfying (a), (b), (c), then there is a proximity relation on X for which $\mathfrak{C}$ is the family of all clusters.
M. E. Shanks (Chapel Hill, N.C.)

7108:

Weston, J. D. A generalization of Ascoli's theorem. Mathematika **6** (1959), 19–24.

Let X and Y be topological spaces, Y^X the space of all functions from X into Y equipped with the topology of uniform convergence on compact sets of X ('compact' here does not imply the Hausdorff separation axiom). The following theorem is proved: If subset $\mathscr{F}$ of Y^X is closed and evenly continuous (for every $x \in X$, $y \in Y$, and neighborhood V of y there are neighborhoods U and W of x and y respectively such that for all $f \in \mathscr{F}$, if $f(x) \in W$ then $f(U) \subseteq V$) and if $\mathscr{F}(x)$ has compact closure for all $x \in X$, then $\mathscr{F}$ is compact. Since equicontinuous families relative to a uniform structure are evenly continuous, the theorem implies Ascoli's Theorem. Tihonov's Theorem follows from the special case where X is discrete, Y compact, and $\mathscr{F}=Y^X$. The Axiom of Choice is also implied by the theorem.
S. Warner (Durham, N.C.)

7109:

Ponomarev, V. Axioms of countability and continuous mappings. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 127–134. (Russian. English summary)

Author's summary: "The following theorems are proved. (1) T_0-spaces are continuous open images of metric spaces if and only if they satisfy the first axiom of countability. The necessary and sufficient condition for the T_0-space X to be an image of a metric space under an open S-mapping

[A. H. Stone, Proc. Amer. Math. Soc. **7** (1956), 690–700; MR **19**, 299] is the existence in X of a point-countable base. In both cases the metric space, the image of which is X, may be supposed zero-dimensional and to have the same weight as the space X. (2) Open S-mappings preserve the property of possessing a point-countable base. (3). If f is a closed continuous mapping of the T_1-space X with a point-countable base onto the T_1-space Y and $f^{-1}(y)$ is a compactum for every $y \in Y$, then Y possesses a point-countable base. From Theorem 1 results the theorem of Schwartz [A. S. Švarc, Uspehi Mat. Nauk **12** (1957), no. 4 (76), 215; MR **19**, 668] and from Theorem 2 and the known results of P. S. Alexandroff follows a theorem of Stone [loc. cit., Th. 4]. (4) A paracompact space which is the image of a metric space under a compact open mapping is metrisable."

[See also Ponomarev, Uspehi Mat. Nauk **14** (1959), no. 4 (88), 203–206; MR **22** #2982.]

R. Arens (Los Angeles, Calif.)

7110:

Church, Philip T.; Hemmingsen, Erik. Light open maps on n-manifolds. Duke Math. J. **27** (1960), 527–536.

For any mapping f of an n-manifold M into an n-manifold N let B_f denote the branch set of f, i.e., the set of all points of M where f fails to be a local homeomorphism. The authors study and analyze light open mappings in this setting with reference to their branch sets. Among other results are the following. (1) If M and N are both E^n, f is open and onto, B_f is compact and $\dim f(B_f) \leq n-2$, $n \neq 2$, and the restriction of f to $M - f^{-1}f(B_f)$ is a covering map, then f is topological. (2) If f is light and open, $\dim B_f = m \leq n-2$ and $\dim f(B_f) < n$, then each point inverse consists of isolated points and $\dim f^{-1}f(B_f) = m$. (3) If M and N are subsets of E^n, f is light, open and C', then either each point inverse consists of isolated points or $\dim B_f = n-1$. (4) If f is light and open, then for no open set U in N can $U \cdot f(B_f)$ be a tamely imbedded $(n-1)$-manifold. (5) If f is light and open and $0 \leq \dim f(B_f) \leq n-2$. then $N - f(B_f)$ has non-trivial local 1-homotopy at each point of $f(B_f)$. The authors conjecture that for $n \geq 3$ there is no light open mapping f with $\dim B_f = 0$ and $\dim f(B_f) < n$ and, in particular, that for $n=3$ there is no light open f for which $\dim f(B_f) = 0$. In this paper they have made considerable progress toward determining the solution to this and related baffling problems concerned with light open mappings on higher dimensional manifolds.

Minor corrections: page 528, line 9 from bottom, $\overline{U}$ should be $\overline{U}'$ in both cases; page 529, references in the proof of (1.4) should be [9; 147, 7.2] and [9; 148, 7.5] respectively.

G. T. Whyburn (Charlottesville, Va.)

7111:

Kwun, Kyung Whan; Raymond, Frank. Generalized cells in generalized manifolds. Proc. Amer. Math. Soc. **11** (1960), 135–139.

The underlying motive of the paper is the reviewer's generalization, in terms of generalized manifolds (gm), of the Schoenflies extension theorem [*Topology of manifolds*, Amer. Math. Soc., New York, 1949; MR **10**, 614; p. 312]. The authors show that if M is a (locally) orientable n-gm with orientable boundary B such that each component B_i of B is a spherelike $(n-1)$-gcm, then the space M^* obtained by identifying each B_i to a point is a (locally) orientable n-gm. The proof utilizes cones over the sets B_i, and as a kind of converse it is shown that if X is a locally orientable n-gm, and $x \in O \subset X$, where O is open with compact closure and boundary O' such that there exists a homeomorphism $h: \overline{O} \to B \vee p$, a cone, with $h(O') = B$ and $h(x) = p$, then $\overline{O}$ is a generalized n-cell. As an important corollary of the latter theorem, one has that a necessary and sufficient condition that a separable metric space X be a classical 3-manifold is that X be a 3-gm and for each $x \in X$ there exist a compact neighborhood which is homeomorphic to a cone over the boundary of the neighborhood. These results utilize theorems for gm's defined over a field. In the concluding sections, the authors extend the latter (and hence the results of this paper) to gm's defined over the group Z of integers. This is done by showing that if X is clc over Z, $\dim_Z(X) < \infty$, and X is an orientable n-gm over the rationals as well as over the integers mod a prime p for all p, then X is an orientable n-gm over Z.

R. L. Wilder (Ann Arbor, Mich.)

7112:

Burago, Yu. D.; Zalgaller, V. A. Polyhedral embedding of a net. Vestnik Leningrad. Univ. **15** (1960), no. 7, 66–80. (Russian. English summary)

Let C be an abstract connected two-dimensional complex originating from a finite number of polygonal regions in the euclidean plane by identifying pairs of (entire) sides of equal length. Then C carries a natural metric, distance of two points being defined as the length of a shortest curve connecting the points. The authors prove the interesting theorem: If C is homeomorphic to a closed region on an orientable surface, then C can be isometrically embedded in E^3 as a polyhedron without self-intersection.

H. Busemann (Los Angeles, Calif.)

ALGEBRAIC TOPOLOGY

See also 6840, 6842, 7102, 7111, 7137, 7141.

7113a:

Grötzsch, Herbert. Zur Theorie der diskreten Gebilde. V. Beziehungen zwischen Vierkant- und Dreikantnetzen auf der Kugel. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 7 (1958), 353–358.

7113b:

Grötzsch, Herbert. Zur Theorie der diskreten Gebilde. VI. Ein Kantentransformationssatz für gerade Dreikantnetze mit Viereckssystem auf der Kugel. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 7 (1958), 447–456.

7113c:

Grötzsch, Herbert. Zur Theorie der diskreten Gebilde. VII. Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 8 (1958/59), 109–120.

7113d:

Grötzsch, Herbert. Zur Theorie der diskreten Gebilde. VIII. Transformation modulo 2 und modulo 3 von Netzen. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 8 (1958/59), 337–344.

7113e:

Grötzsch, Herbert. Zur Theorie der diskreten Gebilde. IX. Über Heawoodsche Gleichungen und Möglichstgleichverteilung von Signaturen. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 8 (1958/59), 747–754.

Papers I–IV of this series [same Z. 5 (1955/56), 839–844; 6 (1956/57), 697–704, 785–788, 789–798; MR **22** #2987a, b, c, d] have already been reviewed and the reader is referred to that review for basic terminology and results. Enough detail has been given there to give some idea of the flavor and techniques of the series. Accordingly this review of papers V–IX will be somewhat less detailed.

V. Relation between four- and three-edged nets on the sphere. A further vertex signature theorem is established concerning 4-nets on the sphere and this leads to a necessary and sufficient condition that a 4-net be transformable to an odd 3-net $N_3(2, 1)$. The paper is largely devoted to the study of two "reduction" operations used in the proof of the theorem.

VI. An edge-transformation theorem for even 3-nets with four-vertex systems on the sphere. This extends the theorem of III to even 3-nets with completely opposite systems of regions (not defined in this review) each region of which is a quadrangle. The theorem is rather involved and the reader is referred to the paper for details.

VII. A three-color theorem for triangle-free nets on the sphere. Theorem: The vertices of a triangle-free net on the sphere are colorable in three colors. The author comments that the theorem has long been known to him. It should be observed that absence of all 3-cycles (not only bounding 3-cycles) is assumed.

VIII. Mod 2 and mod 3 transformations of nets. Theorem: Each 4-edge net or Euler net of a closed orientable or non-orientable surface T may by appropriate "decomposition" of each of its vertices into three-edge vertices be transformed into an even 3-net $N_3(2, 0)$. The appropriate decomposition referred to is a transformation of a 4-edge net which may be described roughly as follows: A four-edge vertex is "split" into two by taking two of the original edges with each, and a new edge is inserted making the two new vertices 3-edge vertices.

IX. Concerning Heawood's equality and the uniform distribution of signatures. Theorem: Suppose T an arbitrary closed orientable or non-orientable surface, N_3 an arbitrary regular or singular 3-net on T for which the system RN_3 of its vertices and edges is either connected or consists of several components. If N_3 contains odd regions (necessarily an even number) and if these are paired in an arbitrary manner, then each vertex E of N_3 may be assigned a Heawood signature $+1$ or -1 in such a way that for each region Γ of N_3, the sum $\Sigma(\Gamma)$ of the signatures $\sigma(E)$ of its boundary vertices, each counted with its appropriate multiplicity, is either ± 1 or zero according as Γ is an odd or an even region. The assignment will also be such that for two paired odd regions $\Sigma(\Gamma)$ will be $+1$ for one of the pair and -1 for the other.

L. M. Kelly (E. Lansing, Mich.)

7114:

Baladze, D. O. Homology and cohomology groups over a pair of coefficient groups. Dokl. Akad. Nauk SSSR **131** (1960), 1234–1237 (Russian); translated as Soviet Math. Dokl. **1**, 401–404.

Let (G, G') and (H, H') be two pairs of compact or discrete groups such that G' and H' are subgroups of G and H respectively, G and H are dual and G' and H' are the annihilators of each other. In the paper homology and cohomology groups of a topological space X with coefficients in a pair of groups are defined in such a way that the groups are compact or discrete according as the coefficient groups are compact or discrete and such that $H_q(X; G, G')$ and $H^q(X; H, H')$ are dual. The use of pairs of coefficient groups is a device which generalizes the notion of homology theory based on finite or infinite chains.

If K is a locally finite complex, $\bar{C}^q(K; G, G')$ denotes the set of all functions u from the q-simplexes of K to G such that $u(s) \in G'$ for almost all s ($G'=0$ and $G'=G$ give respectively finite and infinite cochains). $C^q(K; G, G') = \bar{C}^q(K; G, G')$ when G is discrete and is its completion, with respect to a suitable topology, when G is compact. With the group of chains defined in exactly the same way, $H_q(K; G, G')$ and $H^q(K; H, H')$ are dual. Homology and cohomology groups of a space X are obtained by first taking a limit of all star-finite subcomplexes of the nerve of a given open cover of X and then a limit over all open covers of X.

E. H. Brown (Waltham, Mass.)

7115:

Hirsch, Guy. Sur la définition d'opérations cohomologiques d'ordre supérieur au moyen d'une suite spectrale. Bull. Soc. Math. France **87** (1959), 361–382.

Starting with the cochains C of a space X with coefficients in a field L and an L-algebra $\bar{T}$ of operations on C, the author describes a method of constructing a spectral sequence whose successive differentiation operators are higher order cohomology operations. When $\bar{T}$ is the algebra of Steenrod operations, the method is very similar to that of Adams [Bull. Amer. Math. Soc. **64** (1958), 279–282; MR **20** #3539]. The author shows how the triple product of Massey, the Adem secondary operations, and the functional cup products are special cases of his method. The author also shows how similar constructions can be used to describe the cohomology of the total space of a fibre space; in particular, he obtains a generalization of the Gysin sequence when the fibre is not necessarily a homology sphere.

F. P. Peterson (Oxford)

7116:

Dold, Albrecht. Sur les opérations de Steenrod. Bull. Soc. Math. France **87** (1959), 331–339.

An important problem in algebraic topology is the calculation of the cohomology $H^*(A, q; G)$ of an Eilenberg-MacLane space $K(A, q)$ in terms of cohomology operations for which explicit formulas are known. This paper attacks the problem by using the fact that if $M=M(A, q)$ is an FD-module with only one non-vanishing homology group A in dimension q, then $H^*(A, q; G)$ is canonically isomorphic to $\sum_{n=0}^{\infty} H^*(\mathrm{SP}^n M, G)$, where $\mathrm{SP}^n M$ is the n-fold symmetric product of M.

Let $u \in H^q(X, A)$, where X is an FD-module and A is a finitely generated Abelian group. Steenrod defined the set of nth reduced powers of u in $H^*(X, G)$ as the image of a certain homomorphism, depending on u, A and G. The author modifies this definition so that the homomorphism Φ, whose image is the set of nth reduced powers of the basic class $\beta \in H^q(A, q; A)$, may be written $\Phi = \Psi \circ \Phi'$,

where $\Psi: H^*(C^*(M^n, G)/S_n) \to H^*(A, q; G)$, and Φ' is an epimorphism. Here S_n is the symmetric group, acting on the n-fold Cartesian product M^n by permuting factors. An epimorphism $N^{**}: H^*(C^*(M^n, G)/S_n) \to H^*(C^*(M^n/S_n, G)) = H^*(\mathrm{SP}^n M, G)$ is defined such that the composition of N^{**} with the inclusion $H^*(\mathrm{SP}^n M, G) \to H^*(A, q; G)$ differs from Ψ at most by elements of lower filtration, i.e., of $\sum_{i=0}^{n-1} H^*(\mathrm{SP}^i M, G)$.

These facts imply that every element of $H^*(A, q; G)$ is obtainable from β by reduced powers. Since the reduced powers may be expressed in terms of the elementary operations, the Steenrod powers, and the Pontrjagin-Thomas powers [Steenrod and Thomas, Comment. Math. Helv. **32** (1957), 129–152; MR **19**, 1070], all cohomology operations may be so expressed, at least if the initial coefficient group A is finitely generated. This theorem has also been obtained by J. C. Moore [unpublished], using constructions after the fashion of H. Cartan.

Proofs are omitted. *W. D. Barcus* (Providence, R.I.)

7117:

Wall, C. T. C. Generators and relations for the Steenrod algebra. Ann. of Math. (2) **72** (1960), 429–444.

Let A_2 denote the Steenrod algebra mod 2. It was shown some years ago by J. Adem that the set of Steenrod squares Sq^n for n a power of 2 is a set of generators of A_2. However, it has been an open question to determine a complete set of relations on these generators. In the present paper the author determines such a complete set of relations. The proof that they are a complete set is rather formidable, as is to be expected.

The author observes that this set of generators and relations for A_2 is minimal, and shows how this fact may be used to determine the homology groups $H_1(A_2, Z_2)$ and $H_2(A_2, Z_2)$ (in the sense of homological algebra). The determination of $H_2(A_2, Z_2)$ is originally due to J. F. Adams [Comment. Math. Helv. **32** (1958), 180–214; MR **20** #2711], who found it by consideration of a family of spectral sequences. Finally, the author uses his results to prove some conjectures of H. Toda [Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 33–64; MR **20** #7263] about the algebra A_2.

W. S. Massey (New Haven, Conn.)

7118a:

Chow, Sho-kwan. The Steenrod operations and homotopy groups. I, II. Sci. Record (N.S.) **2** (1958), 355–357, 358–363.

7118b:

Chow, Sho-kwan. Steenrod's operations and homotopy groups. I, II. Acta Math. Sinica **9** (1959), 227–263. (Chinese. English summary)

7118c:

Chow, Sho-kwan. Steenrod operations and homotopy groups. I, II. Sci. Sinica **9** (1960), 155–171, 172–196.

[(c) is the English version of (b).]

In (a)I the author takes a space X which is $(n-1)$-connected (mod p) and determines the p-component of the homotopy group $\pi_m(X)$ for $m \leq \min(2n-2, n+4p-6)$ in terms of homology groups and operations. The phenomena which arise are essentially (i) the Steenrod operation P^1 and (ii) torsion. In (a)I the author gives no proofs, but these are supplied in (c)I; similarly for (a)II and (c)II.

In (a)II the author takes a space which is $(n-1)$-connected $(n>3)$, and undertakes to calculate the 2-component of $\pi_{m+2}(X)$. This would overlap with previous work of K. Shiraiwa [Amer. J. Math. **76** (1954), 235–251; MR **15**, 458]. However, the author claims that he has a counter-example to Shiraiwa's results, namely a pair of A_n^3 complexes X_1, X_2, with different homotopy groups π_{n+2}, which cannot be distinguished homologically except by introducing secondary operations over and above the ones used by Shiraiwa. In the reviewer's opinion, the present author is correct about this example. The construction is as follows. Let $Y = (S^n \cup E^{n+1}) \vee S_1^{n+2} \vee S_2^{n+2}$, where the attaching map for E^{n+1} is of degree 2. Then $\pi_{n+2}(Y)$ has generators β, ι_1, ι_2 arising from the three summands, and β is of order 4. We obtain X_1 from Y by attaching two $(n+3)$-cells, the attaching maps lying in the classes $2^q\iota_1 - \beta$, $2^p\iota_2 - 2\beta$. We obtain X_2 from Y by attaching two $(n+3)$-cells, the attaching maps lying in the classes $2^q\iota_1 - \beta$, $2^p\iota_2$. If the parameters p and q satisfy $p > q+1 \geq 2$, we have

$$\pi_{n+2}(X_1) = Z_{2^{p+1}} + Z_{2^{q+1}},$$
$$\pi_{n+2}(X_2) = Z_{2^p} + Z_{2^{q+2}}$$

(the original has a misprint here).

In view of this example, the obvious next step is to introduce further operations. All the operations in question are variants of Adem's secondary operation Φ; however, the domain of definition of the operation, and consequently its indeterminacy, can be varied significantly; it is desirable to have such things correct. We may proceed as follows.

Let $x \in H^n(K; Z_2)$, $y \in H^{n+2}(K; Z_{2^q})$ be classes such that

$$\mathrm{Sq}^2 x = 0, \quad \mathrm{Sq}^2\,\mathrm{Sq}^1 x = \delta y,$$

where δ is the Bockstein coboundary associated with the exact sequence

$$0 \to Z_2 \to Z_{2^{q+1}} \to Z_{2^q} \to 0.$$

Then the relations

$$\mathrm{Sq}^2\,\mathrm{Sq}^2 = \mathrm{Sq}^1(\mathrm{Sq}^2\,\mathrm{Sq}^1), \quad \mathrm{Sq}^1\,\delta = 0$$

permit one to define an operation Φ^q such that

$$\Phi^q(x, y) \in H^{n+3}(K; Z_2)/(\mathrm{Sq}^2 H^{n+1}(K; Z_2) + \mathrm{Sq}^1 H^{n+2}(K; Z_2)).$$

This operation will serve to distinguish X_1 and X_2.

However, this operation is apparently not the one considered by the author. Unless the details are misprinted, the domain of definition of his operation φ^{q+1} is smaller than that given above; indeed, it is so small that in X_1 and X_2 it becomes zero!

Beyond this, it seems to the reviewer that the indeterminacy of the operation φ^{q+1} is also unrealistically small. The indeterminacy which the author gives does not appear to contain the subgroup $\mathrm{Sq}^1 H^{n+2}(K; Z_2)$; but it can be made plausible that any secondary operation which serves to distinguish between X_1 and X_2, and whose domain of definition depends only on q, will have an indeterminacy containing this subgroup. For, put $p=1$; then the values of $\pi_{n+2}(X_1)$, $\pi_{n+2}(X_2)$ given above are no longer valid, and indeed X_1, X_2 are equivalent, for we can construct a map

from X_2 to X_1 by mapping $(S^n \cup E^{n+1}) \vee S_1^{n+2}$ identically and S_2^{n+2} with class $\iota_2-\beta$. It follows that when we decrease p to be 1, the indeterminacy increases to include the former values of the operation; but the only new primary operation we have introduced is Sq^1.

The doubts raised above may excuse us from examining the rest of the details. It should be said, however, that the general plan and direction of the work seem perfectly sound, and it can presumably be saved.

J. F. Adams (Cambridge, England)

7119:

Barcus, W. D. The stable suspension of an Eilenberg-MacLane space. Trans. Amer. Math. Soc. **96** (1960), 101–114.

This work is related to a previous paper by Barcus and Meyer [Amer. J. Math. **80** (1958), 895–920; MR **20** #5478]. To quote from the introduction to the present paper, "... we set up a spectral sequence for the stable homotopy groups of a countable CW complex X, using the suspension triad sequences for the iterated suspensions S^rX. If X is $(n-1)$-connected, we calculate the differential operator d^1 for the first $n-1$ nontrivial dimensions in terms of the automorphism T_* of $\pi_m(X\times X)$ induced by the map which interchanges factors. ... For $K(\pi, n)$, π finitely generated, we calculate the automorphism T_* on the p-primary component $(p\neq 2)$ for a range of dimensions. The p-primary component $(p\neq 2)$ of the first $n-1$ nontrivial stable homotopy groups can then be read off from the spectral sequence. The corresponding Postnikov invariants are zero, in contrast to those of the single suspension $SK(\pi, n)$; however, this is not true of the 2-primary component."

I. M. James (Oxford)

7120:

Barratt, M. G.; James, I. M.; Stein, N. Whitehead products and projective spaces. J. Math. Mech. **9** (1960), 813–819.

It has long been known how to express the homotopy groups of the real, complex, or quaternionic projective spaces in terms of the homotopy groups of spheres. In this note the authors give expressions for the Whitehead products in similar terms. Most of their results are too complicated to state here; an exception is the case of n-dimensional complex projective space $P_n(C)$ for n odd. In this case $[\alpha, \beta]=0$ if either α or β belongs to $\pi_2(P_n(C))$, and the Whitehead products are completely determined by those in S^{2n+1} via the fibre map $S^{2n+1}\to P_n(C)$. In particular, all Whitehead products vanish in $P_3(C)$, although it is not an H-space. In case n is even, the Whitehead product of the generators of $\pi_2(P_n(C))$ and $\pi_{2n+1}(P_n(C))$ is non-zero.

W. S. Massey (New Haven, Conn.)

7121:

Lima, Elon L. The Spanier-Whitehead duality in new homotopy categories. Summa Brasil. Math. **4**, 91–148 (1959).

The Whitehead-Spanier duality theorem of S-theory says that for subpolyhedra X, $Y\subset S^n$, $\{X, Y\}\approx\{S^n-Y, S^n-X\}$; it is easy to see that this need not be true for more general subsets of S^n. Using spectra, the author defines new groups, in terms of which a result containing the Whitehead-Spanier duality theorem, and valid also for arbitrary compact X, $Y\subset S^n$, is established.

In every spectrum $\mathfrak{L}=(L_i, \lambda_i)$, $i=0, 1, \cdots$, the L_i are finite CW complexes and the connecting maps λ_i are S-maps. (Notation: $\lambda_i: L_i\to L_{i+1}$, $\lambda^i: L_{i+1}\to L_i$.) For direct spectra $\mathfrak{A}=(A_i, \alpha_i)$, $\mathfrak{B}=(B_i, \beta_i)$, define

$$\{\mathfrak{A}, \mathfrak{B}\}_D = \text{Lim Inv}_i[\text{Lim Dir}_j\{A_i, B_j\}];$$

for inverse spectra $\mathfrak{X}=(X_i, \xi^i)$, $\mathfrak{Y}=(Y_i, \eta^i)$ set

$$\{\mathfrak{X}, \mathfrak{Y}\}_I = \text{Lim Inv}_j[\text{Lim Dir}_i\{X_i, Y_j\}].$$

For a finite polytope P, the notation $\{P, \mathfrak{B}\}_D$ [resp. $\{\mathfrak{X}, P\}_I$] means that P is regarded as a direct [resp. inverse] spectrum in evident fashion. A direct $\mathfrak{A}$ and an inverse $\mathfrak{X}$ are called n-dual if for each i, A_i and X_i are weakly n-dual and the weak duality $\{A_i, A_{i+1}\}\approx\{X_{i+1}, X_i\}$ carries α_i to ξ^i. The Whitehead-Spanier duality theorem immediately gives a spectrum duality: if $\mathfrak{A}$, $\mathfrak{X}$ and $\mathfrak{B}$, $\mathfrak{Y}$ are n-dual spectra, $\{\mathfrak{A}, \mathfrak{B}\}_D\approx\{\mathfrak{Y}, \mathfrak{X}\}_I$.

This machinery is applied as follows: A space A is called directly represented if there is a direct spectrum $\mathfrak{A}$ and an S-map $\lambda:\mathfrak{A}\to A$ (i.e., each $\lambda_i: A_i\to A$ is an S-map and $\lambda_i=\lambda_{i+1}\alpha_i$) which induces an isomorphism $\{P, \mathfrak{A}\}_D\approx\{P, A\}$ for every finite polytope P; if $\mathfrak{A}$, $\mathfrak{A}'$ [resp. $\mathfrak{B}$, $\mathfrak{B}'$] directly represent A [resp. B], it is shown that $\{\mathfrak{A}, \mathfrak{B}\}_D\approx\{\mathfrak{A}', \mathfrak{B}'\}_D$. A space X is inversely represented if there is an inverse spectrum $\mathfrak{X}$ and an S-map $\mu: X\to\mathfrak{X}$ inducing $\{\mathfrak{X}, P\}_I\approx\{X, P\}$ for every finite polytope P; again $\{\mathfrak{X}, \mathfrak{Y}\}_I$ depends only on the inversely represented spaces. Two spaces are called n-dual if they have n-dual representing spectra; this extension of the notion of duality is shown to be useful by establishing that any compact $X\subset S^n$ is inversely representable, and has S^n-X as n-dual. The spectrum duality, restricted to representing spectra, therefore constitutes an extension of the Whitehead-Spanier theorem valid also for compact X, $Y\subset S^n$.

The author also defines a "singular" and a "Čech" S-group for spaces, to which the groups $\{\mathfrak{A}, \mathfrak{B}\}_D$ and $\{\mathfrak{X}, \mathfrak{Y}\}_I$ reduce whenever the spectra involved are representing. Let $\mathfrak{P}$ be the class of all finite polytopes. For spaces A, B, the "singular" S-group $\{A, B\}_S$ is the subgroup of the direct product $\prod_{P\in\mathfrak{P}}\text{Hom}[\{P, A\}, \{P, B\}]$ consisting of all those elements $\sigma=[\sigma_L]$ whose coordinates σ_L satisfy the conditions: for any S-map $f: P\to Q$ of finite polytopes, $\sigma_P f^A=f^B\sigma_Q$ ($f^Y:\{Q, Y\}\to\{P, Y\}$ is the induced homomorphism); if A, B are directly represented by $\mathfrak{A}$, $\mathfrak{B}$ then $\{\mathfrak{A}, \mathfrak{B}\}_D\approx\{A, B\}_S$. Similarly, the subgroup of $\prod_{P\in\mathfrak{P}}\text{Hom}[\{Y, P\}, \{X, P\}]$ obtained by imposing the analogous coordinate condition relative to S-maps of finite polytopes is called the "Čech" S-group $\{X, Y\}_C$, and is $\approx\{\mathfrak{X}, \mathfrak{Y}\}_I$ whenever $\mathfrak{X}$, $\mathfrak{Y}$ inversely represent X, Y.

Among other results established in this paper is: A space A has an n-dual A^*, with A^* inversely represented, if and only if the singular homology $H_*(A)$ is countable and bounded.

J. Dugundji (Los Angeles, Calif.)

7122:

Kan, Daniel M. Homotopy groups, commutators, and Γ-groups. Illinois J. Math. **4** (1960), 1–8.

Let F be a connected semi-simplicial group-complex. We define the "commutator complex" $[F, F]\subset F$ by $[F, F]_n=[F_n, F_n]$. Let K be a reduced (i.e., with only one vertex) semi-simplicial complex and GK the "loop-space" in the sense of the author in Ann. of Math. (2) **67** (1958), 282–312 [MR **22** #1897]; there he proved that

$$\pi_n(K) = \pi_{n-1}(GK).$$

Now he defines

$$\gamma_n(K) = \pi_{n-1}([GK, GK]).$$

This group is intimately related to the group

$$\Gamma_n(K) = \text{image } (\pi_n(K^{n-1}) \to \pi_n(K^n)),$$

introduced by Whitehead [ibid. **52** (1950), 51–110; MR **12**, 43]. Indeed, the following is proved. Let K be simply connected; then there is a commutative diagram

$$\begin{array}{ccccccccc} \to & H_{n+1}(K) & \to & \Gamma_n(K) & \to & \pi_n(K) & \to & H_n(K) & \to \\ & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow{\scriptstyle \psi_n} & & \| & & \downarrow{\scriptstyle \alpha_n} & \\ \to & \pi_n(AK) & \to & \gamma_n(K) & \to & \pi_n(K) & \to & \pi_{n-1}(AK) & \to \end{array}$$

where AK is GK "made abelian", α_{n+1} and α_n are isomorphisms [cf. Kan, op. cit.] and ψ_n is an isomorphism; the top sequence is the exact sequence introduced by Whitehead [op. cit.]. The author observes that α_n, ψ_n are dual to each other in the categorical sense.

V. Gugenheim (Baltimore, Md.)

7123:

Connor, P. E.; Dyer, Eldon. On singular fiberings by spheres. Michigan Math. J. **6** (1959), 303–311.

A singular fibration is a quadruple $\{(X, A), (Y, B), \pi, F\}$, with π a proper open map of (X, A) onto (Y, B), $\pi|(X-A)$ a fibering over $Y-B$ with fiber F, and $\pi|A$ a homeomorphism with B. The authors assume F to be a (cohomology) r-sphere; all homology is over Z_2. The main tool is the Gysin sequence for $X-A$, with a new argument (for each compact cohomology class in $Y-B$ the cup product with some power of the characteristic class is 0). The results resemble those for the stationary set of a transformation group. E.g.: If X is (compact and) acyclic, so are A and Y; and a similar local statement. If X is an n-sphere, then A is an $(n-k(r+1))$-sphere for some k; if X is a compact, strongly paracompact, locally orientable generalized n-manifold, then the components of A are also of this type, with dimensions $\equiv n \pmod{r+1}$.

H. Samelson (Princeton, N.J.)

7124:

Weier, Josef. Homologiebedingungen zweiter und dritter Ordnung für die Wesentlichkeit einer Abbildung. Monatsh. Math. **64** (1960), 39–50.

Let P and Q denote orientable triangulated manifolds of dimension m and n respectively, $m>n$. The author studies sufficient conditions for a map $f: P\to Q$ to be essential. Following his earlier papers [Collect. Math. **10** (1958), 45–58, 59–68; MR **21** #328, 329] the author associates with each pair (f, a), $a\in Q$, a finite sequence of cycles z_i (called "solution cycles") with coefficients in $\pi_{m-r-1}(S^{n-1})$; here r denotes dimension of the polyhedron $f^{-1}(a)$, $r\leq m-n$. f is essential whenever at least one z_i fails to be zero-homologous. These are the "second order homology conditions" referred to in the title. "Third order conditions" occur when $m=2n-1$, $r=n-1$, and $z_i\sim 0$ for all i. Then linking numbers ω_{jk} of z_j and z_k are considered. If at least one $\omega_{jk}\neq 0$, f is essential.

S. Mardešić (Zagreb)

7125:

Brody, E. J. The topological classification of the lens spaces. Ann. of Math. (2) **71** (1960), 163–184.

In this paper the author carries out the topological classification of the 3-dimensional lens spaces using a method without reference to the Hauptvermutung which was first proposed by R. H. Fox. Let M be an orientable 3-dimensional manifold, γ an element of $H_1(M)$, k a simple closed curve which represents γ and which is polygonal in some triangulation of M. Let $\Delta(M-k)$ be the Alexander polynomial of k, $T_1(M-k)$ and $B_1(M-k)$ the torsion group and Betti group of $H_1(M-k)$, $i: H_1(M-k)\to H_1(M)$ the injection homomorphism, and $*: B_1(M-k)\to H_1(M)/iT_1(M-k)$ the homomorphism induced by i. First he proves: (i) $H_1(M-k)$ and $iT_1(M-k)$ depend only upon γ, (ii) $\Delta^*(M-k)$ depends only upon γ (so may be denoted as $\Delta^*(\gamma)$). Then, characterising the homology classes γ of a lens space L having the properties: (1) $H_1(L-\gamma)=Z$; (2) $\Delta^*(\gamma)=1$, he shows that the lens space $L(p, q)$ and $L(p, q')$ are homeomorphic if and only if either $q\equiv\pm q'$ or $qq'\equiv\pm 1 \pmod p$. As an additional example he gives a classification of the topological sums of two 3-dimensional lens spaces. The invariant Δ^* is applicable to all orientable 3-dimensional manifolds. As possibilities for further applications of the Δ^* invariant he mentions: (i) the manifolds constructed by Ausbohrung of knots in S^3, (ii) the "fibre space" of Seifert, and (iii) Alexander's theorem that every 3-dimensional manifold is a branched covering of a multiple knot in S^3.

J. Tao (Osaka)

7126:

Mal'cev, A. A. A duality theorem for non-closed sets in manifolds. Dokl. Akad. Nauk SSSR **126** (1959), 709–712. (Russian)

Continuing the work of Sitnikov [Dokl. Akad. Nauk. SSSR **96** (1954), 925–928; Mat. Sb. (N.S.) **34 (76)** (1954), 3–54; MR **17**, 70; **16**, 736], the author announces a duality theorem for subsets of orientable manifolds which are acyclic in certain dimensions. He includes a sketch of the proof and indicates an application to a theory of dimension.

D. W. Kahn (New Haven, Conn.)

7127:

Tynyanskiĭ, N. T. Extension of the K. A. Sitnikov duality law to the case of sets lying in manifolds which do not satisfy the acyclicity conditions. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 21–31. (Russian)

In this paper true cycles and homology are taken in the strong sense of K. A. Sitnikov [Mat. Sb. (N.S.) **34 (76)** (1954), 3–54; MR **16**, 736]. M^n is a closed orientable homology n-manifold, A an arbitrary subset of M^n and $B=M^n\setminus A$. $\triangle^q(B:M^n)$ denotes the quotient group obtained by reducing the group of all the true q-cycles which have compact carriers in B and are zero-homologous in M^n, modulo the subgroup of true q-cycles which are zero-homologous in B. $\nabla^p(A:M^n)$ denotes the quotient group of the Čech p-cohomology group ∇^pA (based on star-finite open coverings of A) reduced modulo the subgroup $\nabla^p_{\text{sect}}(A:M^n)$, defined as follows. Each cocycle z_γ of a finite covering γ of M^n defines an excised cocycle $z_{\gamma A}$ belonging to the intersection of γ by A; z_γ is said to prolong $z_{\gamma A}$ in M^n. $\nabla^p_{\text{sect}}(A:M^n)$ is the subgroup of $\nabla^p(A)$ generated by the prolongable p-cocycles of A.

Theorem: $\triangle^q(B:M^n)\approx\nabla^p(A:M^n)$, for $p+q=n-1$; both groups are taken with an arbitrary coefficient group $\mathfrak{G}$. In the case when M^n is acyclic in dimensions q and $q+1$, this result reduces to the Sitnikov duality law [op. cit.].

S. Mardešić (Zagreb)

7128a:

Liao, S. D. A note on local products and duality of Poincaré-Alexander-Lefschetz type. Acta Math. Sinica **7** (1957), 183–199. (Chinese. English summary)

7128b:

Liao, S. D. On local products and duality of Poincaré-Alexander-Lefschetz type. Sci. Sinica **6** (1957), 977–993.

Chinese and English versions of the same paper, which was written in 1956. For recent literature on the subject see #7129 below. A compact Hausdorff space X is called n-manifold-like over a coefficient field F if at least one of three statements (a), (b), (c) is true. The statement (b) is as follows: there is an element $S(X)$ in $H_n(X)$ such that $\frown S(X): H^i(X_0|X) \to H_{n-i}(X_0)$ is an isomorphism for every i and every closed set X_0 in X; coefficients are in F. The second H is the projective limit of $(H_{n-i}(X-W))$, where W is an arbitrary open neighborhood of X_0. Statements (a) and (c) are similar. The three statements are equivalent when X is connected and finite-dimensional and the local Betti numbers $p^i(x, X)$ are $\leq \omega$ for every i and x. The three statements are true when X is a triangulable, connected orientable n-manifold. Statement (b) holds if X is connected and dim $H_i(x, X)$ is 1 for $i=n$ and 0 for $i\neq n$, and $H_n(X^1)=0$ for every proper closed subset X^1 of X ($H_i(x, X)$ being defined as a direct limit).

P. A. Smith (New York)

7129:

Borel, Armand. ★Seminar on transformation groups. With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Annals of Mathematics Studies, No. 46. Princeton University Press, Princeton, N.J., 1960. vii+245 pp. $4.50.

The transformation groups considered in this volume are topological transformation groups (G, X) in which G is a compact Lie group, possibly zero-dimensional. Generally speaking, attention is focused on such properties of fixed-point sets, orbits and orbit spaces as can be described by homology theory. The space X on which G acts is frequently assumed to have simple homological properties, globally or locally or both. For example, X frequently belongs to the class of "cohomology manifolds", the properties of which are developed in the first two chapters. There follows a study of the action (Z_p, X), p a prime, first in a chapter by Floyd using exact sequences of cohomology groups over sheaves, then by Borel using spectral sequences. The basic idea in Borel's treatment is the use of the space X_G which is the orbit space, under diagonal action, of $X\times E$, where E is the classifying space of G. The spectral sequences which are used are those of the maps $(X\times E)/G \to X/G$ and $(X\times E)/G\to E/G$ induced by the projections of $X\times E$. By the same technique, Borel extends certain results of Conner and Floyd on actions (T_1, X) where T_1 is the 1-dimensional toral group. The next chapter, which is by Floyd, contains the first stage of the proof of the theorem that if X is a cohomology manifold and C a compact set in X, there exist only a finite number of distinct isotropy subgroups G_x for $x\in C$ (G_x consists of those elements g of G such that $gx=x$). What Floyd does is to prove this for G a toral group; the passage to compact Lie groups is given by Bredon in the chapter which follows. A chapter by Deane Montgomery contains a unified and rather concise account of the decomposition of X into families of orbits. The orbits which constitute a given family are those whose stability groups have certain specific properties such as having a specified dimension or number of components. The main results in this promising field are by Montgomery and his collaborators. An important tool is the "slice", which receives a very readable treatment in a chapter by Palais devoted to Mostow's theorem that (G, X) can be imbedded equivariantly in a euclidean action (G, R^n) if X and (G, X) satisfy certain conditions of finiteness. Montgomery showed in his chapter that if k is the dimension of the orbits of highest dimension and X is an n-manifold, and if $F(G, X)$ is the fixed-point set (consisting of the points x such that $gx=x$ for all g) then dim $F\leq n-k-1$. Bredon in the second of his two chapters shows that when equality holds, then in the neighborhood of any point x of F the structure of (G, X) has many of the features which are characteristic of differentiable actions: the orbits, for example, have a local cross-section. Borel has devoted several chapters to the spectral sequence of Fary, a generalization of Leray's spectral sequence of a map, and has proved a number of new fixed-point theorems by way of application. He shows, for instance, that when $G=Z_p\times\cdots\times Z_p$, then dim $H^*(F(G, X), K)=$ dim $H^*(X, K)$, K a field of characteristic p, provided that (G, X) satisfies certain conditions which are rather technical but which nevertheless yield interesting special cases. In addition to the theorems on transformation groups, the book contains much useful expository material by Borel on gratings, sheaves, bundles, spectral sequences and so on. As a whole, the book gives an impressive view of modern technique in action.

P. A. Smith (New York)

7130:

Bredon, Glen E. Orientation in generalized manifolds and applications to the theory of transformation groups. Michigan Math. J. **7** (1960), 35–64.

Let M be a generalized cohomology manifold (in the sense of #7129 above), cohomology computed with respect to a principal ideal ring L. Let k denote the order of the automorphism group of L. There is constructed a k-fold covering M^* of M, which is orientable. (If L is Z, M^* is the orientable double covering.) A transformation group G acting on M determines a unique orientation-preserving transformation group G^* acting on M^*.

The remainder of the paper deals with the theory of transformation groups of prime period p acting on a Z_p cohomology manifold M. It is proved that if p is odd, then the dimensions of M and of each non-empty component of the fixed point set are of the same parity (known previously for cohomology spheres). Modern proofs of theorems of P. A. Smith, that the fixed point set is a Z_p cohomology manifold, and orientable if M is orientable, are given, and it is shown how proofs of the preceding theorems may be used to obtain the global Smith theorems. A theorem regarding the position of the fixed point set in the orbit space is proved.

T. E. Brahana (Athens, Ga.)

7131:

Livesay, G. R. Fixed point free involutions on the 3-sphere. Ann. of Math. (2) **72** (1960), 603–611.

It is shown that every transformation group (Z_2, S^3)

which is free of fixed points, is isomorphic to the transformation group (Z_2, S^3) in which Z_2 acts antipodically. It is first shown that S^3 contains an invariant simple closed curve bounding a disc which is imbedded as a subcomplex of a triangulation of S^3 in which the involution T is simplicial. *P. A. Smith* (New York)

7132:

Hadwiger, H. Elementare Kombinatorik und Topologie. Elem. Math. **15** (1960), 49–60.

Based on a combinatorial lemma of A. W. Tucker [Proc. 1st Canadian Math. Congr. (Montreal, 1945), pp. 285–309, Univ. of Toronto Press, Toronto, 1946; MR **8**, 525], four antipodal-point theorems are proved. Theorem 1 contains the classical Lusternik-Schnirelmann theorem and, as is pointed out by the author, is contained in a somewhat more general result of the reviewer [Ann. of of Math. (2) **56** (1952), 431–437; MR **14**, 490]. Theorem 2 is dual to Theorem 1. Theorem 3: Let f_i $(1 \leqq i \leqq k)$ be k real-valued continuous functions on the n-sphere S^n. If, for every point $p \in S^n$, the relation $f_i(p) = f_i(p^*)$ holds for at least $k-n$ distinct indices i, where p^* denotes the antipodal point of p, then there exists a point $q \in S^n$ such that $f_i(q) = f_i(q^*)$ for $1 \leqq i \leqq k$. Theorem 4: If f_i $(1 \leqq i \leqq n-1)$ are $n-1$ real-valued continuous functions on S^n such that $f_i(p) = f_i(p^*)$ for all $p \in S^n$ and $1 \leqq i \leqq n-1$, then there exists a continuum $C \subset S^n$ such that C is invariant under the antipodal-point mapping and each f_i is constant on C. The case $k = n$ of Theorem 3 is, of course, the Borsuk-Ulam theorem. *Ky Fan* (Detroit, Mich.)

7133:

Bognár, M. n-Dimensionale berandete Pseudomannigfaltigkeiten im $(n+1)$-dimensionalen euklidischen Raume. I. Acta Math. Acad. Sci. Hungar. **10** (1959), 363–373. (Russian summary, unbound insert)

Theorem: The cone over a non-orientable closed $(n-1)$-pseudo-manifold, and the product of a non-orientable n-pseudo-manifold with boundary and a non-discrete space, cannot be imbedded in Euclidean $(n+1)$-space E^{n+1}. The proofs are based on another theorem which will be proved in chapter V of the paper: Every non-orientable [resp. orientable, without interior homology-singular points] n-pseudo-manifold with non-empty boundary is absolutely linked [resp. unlinked]. Here absolutely linked means that for every imbedding into E^{n+1} some 1-cycle (mod 2) in the interior is linked with "the" $(n-1)$-cycle on the boundary; absolutely unlinked means that such linking never happens, including the case that no imbedding exists. A series of eight chapters is announced; the last one will prove that a locally compact connected n-dimensional abelian group with countable basis can be imbedded in E^{n+1} if and only if it is the product of a k-torus and E^{n-k}, generalizing a result of Kodaira and Abe [Proc. Imp. Acad. Tokyo **16** (1940), 167–172; MR **2**, 5]. *H. Samelson* (Princeton, N.J.)

7134:

Mazur, Barry. The definition of equivalence of combinatorial imbeddings. Inst. Hautes Études Sci. Publ. Math. **1959**, 97–109.

Let K and K' be combinatorially isomorphic subcomplexes of r-dimensional euclidean space E^r, and $f_t: K \to K'$ be an isotopy between K and K' with respect to some fixed subdivision of K. The author proves that the isotopy f_t can be extended to an ambient isotopy $F_t: E^r \to E^r$ covering f_t such that $F_t|K = f_t$. The construction of the extended isotopy is done in two stages. The first stage is to reduce the problem to the case of simple isotopy which is constant on every vertex of K except for one. The second is to extend the simple isotopy to an ambient isotopy by restating it as a cross-section extension problem of fibre bundles. *H. Noguchi* (Tokyo)

7135:

Mazur, Barry. On the structure of certain semi-groups of spherical knot classes. Inst. Hautes Études Sci. Publ. Math. **1959**, 111–119.

The author gives a classification of pairs (S_1, S_2), where S_1 is a k-sphere in an r-sphere S_2, with respect to a $*$-equivalence, that is, two pairs (S_1, S_2) and (S_1', S_2') are $*$-equivalent if there is an orientation-preserving homeomorphism $\varphi: S_2 \to S_2'$ bringing S_1 onto S_1', provided that φ is combinatorial except at a finite number of points. The set of all $*$-equivalent classes of pairs forms a semi-group $*\Sigma_k^r$, certain semi-groups being defined in the paper. The result is: a locally unknotted pair (S_1, S_2) is $*$-trivial if and only if there is a locally unknotted pair (S_1', S_2') such that the sum of both pairs is trivial in the usual sense. A theorem on infinite sums in $*\Sigma_k^r$ is obtained in the last section. *H. Noguchi* (Tokyo)

7136:

Mazur, Barry. Orthotopy and spherical knots. Inst. Hautes Études Sci. Publ. Math. **1959**, 121–140.

The main result of the paper is: For a broad range of dimensions $r \geqq (3n+5)/2$ any (locally unknotted and homogeneous) n-sphere knot S in E^r is $*$-trivial. (S is homogeneous if for any family of homeomorphisms $P_t: S \to S$ such that $P_0 =$ identity, and for any regular neighbourhood N of S, there is a homeomorphism $P: E^r \to E^r$ such that $P|E^r - N =$ identity, $P|S = P_1$.) The proof is divided into two parts. The first part is devoted to an "orthotopy theorem", i.e., if K and K' are simplicial isomorphic complexes, $\varphi: K \to K'$ in E^r, there is a local isotopy $\varphi_t: K \to E^r$ such that $\varphi_0 =$ identity, $\varphi_1 = \varphi$, and such that the difference between r and the dimension of the subspace spanned by Δ_1 and Δ_2 is $\leqq 1$, if Δ_1 and Δ_2 are distinct simplices in K and φ_t(interior of Δ_1) $\cap$ φ_t(interior of Δ_2) is not empty. In the second part the author describes a "modification" of S in E^r to get the main result, using the orthotopy theorem and theorems obtained in the preceding two papers [#7134, #7135]. *H. Noguchi* (Tokyo)

DIFFERENTIAL TOPOLOGY

See also 7078.

7137:

Srinivasacharyulu, Kilambi. Sur certaines variétés triangulables. C. R. Acad. Sci. Paris **250** (1960), 2316–2317.

The author constructs examples (a) of pairs of manifolds that are homotopy-equivalent but not combinatorially equivalent and (b) of triangulated manifolds of dimension

16 that do not possess compatible differentiable structures. To (a): for $r>8$ the S_{r-1}-bundle over S_8, corresponding to $m \in \pi_7(SO(r))=Z$, has second Pontryagin class $\pm 6m \times$ generator of $H^8(S_8)$. (The factor 6 comes from $\pi_6(SU(3))$.) On the other hand, two such spaces are of the same homotopy type if $m \equiv m'$ (mod 240). To (b): For suitable S_7-bundles over S_8 the Thom space is a triangulated manifold with fractional fourth Pontryagin class. The construction utilizes explicit knowledge of $\pi_7(SO(8))=Z+Z$, and the index formula. *H. Samelson* (Princeton, N.J.)

7138:

Hirsch, Morris W. An exact sequence in differential topology. Bull. Amer. Math. Soc. **66** (1960), 322–323.

For each positive integer n, the author defines abelian groups Γ^n, θ^n, and Λ^n as follows. Γ^n is the group of all diffeomorphisms of the $(n-1)$-sphere S^{n-1} modulo the normal subgroup of those diffeomorphisms that are extendable to the n-ball. θ^n is the set of J-equivalence classes of compact, oriented, differentiable n-manifolds that are homotopy spheres; two such oriented manifolds, M and N, are defined to be J-equivalent if there is an oriented $(n+1)$-manifold X whose boundary is the disjoint union of M and $-N$ and which admits both M and N as deformation retracts. The sum of two J-equivalence classes is defined by forming the sum of two representative manifolds in the usual way. The group Λ^n is defined in an analogous way using combinatorial manifolds instead of differentiable manifolds. Next, the author defines homomorphisms $j:\Gamma^n \to \theta^n$, $k:\theta^n \to \Lambda^n$, and $d:\Lambda^n \to \Gamma^{n-1}$. Here k associates with each J-equivalence class of differentiable manifolds the corresponding equivalence class of combinatorial manifolds determined by a smooth triangulation. The definitions of j and d are more involved. The author asserts as a theorem that the sequence

$$\cdots \to \Gamma^n \to \theta^n \to \Lambda^n \to \Gamma^{n-1} \to \cdots$$

is exact. The proof is promised to appear in a subsequent paper. *W. S. Massey* (New Haven, Conn.)

7139:

Cerf, Jean. Groupes d'automorphismes et groupes de difféomorphismes des variétés compactes de dimension 3. Bull. Soc. Math. France **87** (1959), 319–329.

The author announces and gives an outline of a proof of the following: Let F be a compact differentiable manifold of dimension 3, G the space of topological homeomorphisms of F, with compact open topology, and H the space of diffeomorphisms of F with the C^1 topology. Then the inclusion map of H into G is a homotopy equivalence. The proof depends essentially on a "theorem" of the reviewer to the effect that the inclusion of SO(4) into the space of orientation-preserving diffeomorphisms of S^3 induces an isomorphism of the homotopy groups. The proof of this "theorem" (never published), however, has since been found to be incomplete. Thus, as the author has written the reviewer, in his work, the "theorem of Smale" should be replaced by the "conjecture of Smale" and the abovequoted result of his paper can be read as before, qualified by "if the 'conjecture of Smale' is true". The paper contains other interesting theorems, most of which depend on the "conjecture of Smale". *S. Smale* (Berkeley, Calif.)

7140:

Palais, Richard S. Natural operations on differential forms. Trans. Amer. Math. Soc. **92** (1959), 125–141.

Soit M une variété indéfiniment différentiable de dimension n et Φ^p l'espace des formes différentielles extérieures de degré p indéfiniment différentiables sur M. Tout automorphisme g de M induit un automorphisme g^p de Φ^p. L'Auteur appelle $\mathscr{I}(\Phi^p, \Phi^q)$ le vectoriel des applications linéaires $T:\Phi^p \to \Phi^q$ telles que l'on ait $T\circ g^p = g^q \circ T$ pour tout $g \in G$. Désignons par G le groupe des automorphismes de M et par G_C les automorphismes à support compact (c'est-à-dire, qui laissent fixe le complémentaire d'une partie compacte de M). L'Auteur munit G d'une topologie naturelle qui en fait un groupe topologique et permet de définir G_0, composante connexe par arcs de l'identité de G_C. Cela étant, $\mathscr{I}^0(\Phi^p, \Phi^q)$ désigne le vectoriel des applications linéaires $T_0:\Phi^p \to \Phi^q$ telles que l'on ait $T_0 \circ g_0{}^p = g_0{}^q \circ T_0$ pour tout $g_0 \in G_0$. Enfin, $d^p:\Phi^p \to \Phi^{p+1}$ désignant la restriction de la différentielle extérieure aux formes de degré p, on pose $Z^p=(d^p)^{-1}0$, $B^p=d^{p-1}\Phi^{p-1}$, $H^p=Z^p/B^p$ (les Z^p et B^p étant regardés comme vectoriels réels). La signification de $\mathscr{I}(Z^p, \Phi^q)$ et $\mathscr{I}^0(Z^p, \Phi^q)$ se comprend alors d'elles-mêmes.

L'Auteur établit les propriétés suivantes:

$\mathscr{I}^0(\Phi^p, \Phi^q) = \mathscr{I}(\Phi^p, \Phi^q) = 0$ si $0 \leq p \leq n$, $0<q\leq n$, et $q \neq p, p+1$;

$\mathscr{I}^0(\Phi^p, \Phi^p) = \mathscr{I}(\Phi^p, \Phi^p)$ = vectoriel des homothéties de Φ^p lorsque $p>0$;

$\mathscr{I}^0(\Phi^p, \Phi^{p+1}) = \mathscr{I}(\Phi^p, \Phi^{p+1})$ = vectoriel sous-tendu par d^p;

$\mathscr{I}^0(Z^p, \Phi^q) = \mathscr{I}(Z^p, \Phi^q) = 0$ si $q>0$ et $p \neq q$;

$\mathscr{I}^0(Z^q, \Phi^q) = \mathscr{I}(Z^q, \Phi^q)$ = vectoriel engendré par l'inclusion si $q>0$.

Si la variété M est compacte, l'Auteur précise que

$\mathscr{I}^0(\Phi^p, \Phi^0) = \mathscr{I}(\Phi^p, \Phi^0) = 0$ si $0<p<n$;

$\mathscr{I}^0(\Phi^0, \Phi^0) = \mathscr{I}(\Phi^0, \Phi^0)$ = vectoriel des homothéties de Φ^0;

$\mathscr{I}^0(\Phi^n, \Phi^0)$ = vectoriel engendré par l'intégration sur le cycle fondamental, si M est orientable, et

= 0 si M n'est pas orientable;

$\mathscr{I}(\Phi^n, \Phi^0)$ = 0 si M n'est pas orientable, ou si M est orientable et réversible, et

= vectoriel engendré par l'intégration sur le cycle fondamental si M est orientable et irréversible;

$\mathscr{I}^0(Z^p, \Phi^0) \simeq H_p(M)$;

$\mathscr{I}(Z^p, \Phi^0) \simeq N_p(M)$, le sousvectoriel de $H_p(M)$ comprenant les seuls éléments laissés fixes à gauche par les automorphismes de $H_p(M)$ induits par ceux de M.

G. Papy (Brussels)

7141:

Hirsch, Guy. Sur certaines opérations dans l'homologie des espaces de Riemann. Bull. Soc. Math. Belg. **9** (1957), 115–139.

As is well known, one may use exterior differential forms as cochains on a differentiable manifold to compute the real cohomology ring; this is the content of de Rham's theorem. Moreover if the differentiable manifold has a Riemannian metric, then the vector space of exterior

p-forms has a canonical direct sum decomposition into the harmonic forms, the forms homologous to 0, and the forms cohomologous to 0 [see G. de Rham, *Variétés différentiables*, Actualités Sci. Ind. no. 1222, Hermann, Paris, 1955; MR **16**, 957; p. 157]. Thus every cohomology class contains a unique harmonic form, and every exterior form which is cohomologous to 0 is the coboundary of a unique form which is homologous to 0.

This paper is based on the observation that if one uses exterior differential forms on a Riemannian manifold to compute higher order real cohomology operations, such as the triple product, or functional cohomology operations, such as the functional cup product of Steenrod, then it is possible to define them as unique cohomology classes rather than as cosets of certain cohomology groups. To do this, one imposes two additional conditions in the usual definition of these operations, as follows: (a) Whenever one has to choose a representative cocycle for a cohomology class, choose the unique harmonic form in that class; (b) whenever one has to choose a form which a certain cocycle cobounds, choose the unique such form which is homologous to zero. Of course the unique cohomology operations or functional cohomology operations thus defined depend on the choice of the Riemannian metric.

The author extends this process to define a whole series of new higher order cohomology operations and functional cohomology operations on Riemannian manifolds. For most of these operations, he does not give any applications or examples. As an application of some of them, he shows how they can be used to give a theoretical description of the structure of the real cohomology ring of a sphere bundle over a Riemannian manifold in the case where the fibre is homologous to zero (over the reals). The structure depends on the knowledge of certain of these cohomology operations in the base space. Apparently the success of this method depends on the fact that the multiplication of exterior differential forms is anti-commutative.

{Reviewer's remark: Whenever a field is used as coefficient domain for cohomology, a decomposition of the cochain groups similar to that described above is possible; of course such a decomposition is not canonical. Thus all the procedures outlined in this paper which do not depend on the anti-commutativity of exterior differential forms can be carried through in this more general case.}

W. S. Massey (New Haven, Conn.)

7142:

Elianu, Ion. Courants autoadjoints dans un espace de Riemann non compact et quelques applications. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 239–248.

Sur un espace de Riemann non compact muni d'une structure C^∞, soit $*$ l'opérateur d'adjonction. Un courant T (non nécessairement homogène) est dit autoadjoint si $*T=T$ [les notations et définitions sont celles de de Rham, *Variétés différentiables*, Actualités Sci. Ind. no. 1222, Hermann, Paris, 1955; MR **16**, 957]. Soit $\mathscr{D}_a'$ l'espace vectoriel des courants autoadjoints: $\mathscr{D}_a'$ est complet. Soit $\overline{\mathscr{D}}$ l'espace des courants de carré sommable. Alors: $\overline{\mathscr{D}}_a=\overline{\mathscr{D}}\cap\mathscr{D}_a'$ est un espace de Hilbert, et $\overline{\mathscr{D}}$ est la somme directe de $\overline{\mathscr{D}}_a$ et du sous-espace $\overline{\mathscr{D}}_0$ orthogonal à $\overline{\mathscr{D}}_a$; si $T\in\overline{\mathscr{D}}$ est C^∞, chacune de ses composantes est C^∞. Le théorème de décomposition des courants de $\overline{\mathscr{D}}$ dû à Kodaira [voir ouvrage cité] entraîne: $\overline{\mathscr{D}}_a$ est la somme directe de trois sous-espaces dont les deux premiers sont échangés par $*$ et le troisième est l'espace des formes fermées et cofermées autoadjointes. Ces deux résultats entraînent la décomposition directe de $\overline{\mathscr{D}}$ en quatre sous-espaces. D'autres énoncés sont obtenus en décomposant $\overline{\mathscr{D}}_0$. Enfin ces résultats s'étendent aux courants continus en moyenne à l'infini.

P. Dolbeault (Poitiers)

7143:

Samelson, Hans. On immersion of manifolds. Canad. J. Math. **12** (1960), 529–534.

Let $f:M\to M'$ be an immersion of a compact oriented manifold M into an oriented manifold M'. Let b denote the fundamental homology class of the unit normal bundle B of the immersion, and s' the homology class of the unit tangent bundle T' of M' represented by a fibre. Let $\nu:B\to T'$ be the normal or Gauss map. An elementary proof is given of the relation $\nu_*b=\chi\cdot s'$, where $\nu_*:H_*(B)\to H_*(T')$ is induced by ν, and χ is the Euler-Poincaré characteristic of M. A second proof, using Morse theory, is given for the special case $M'=$Euclidean space.

M. W. Hirsch (Berkeley, Calif.)

7144:

Haefliger, André. Sur les self-intersections des applications différentiables. Bull. Soc. Math. France **87** (1959), 351–359.

Let V be a differentiable manifold of dimension n, and π a subgroup of the group of permutations of p objects. Let $V_0{}^p$ be the subspace of the cartesian product V^p consisting of elements with p distinct coordinates. Let V_π be the orbit space of $V_0{}^p$ under the obvious action of π. If M is a differentiable manifold, $E^r\to V$ is the bundle of jets of order r of V into M, in the sense of Ehresmann, and $E_\pi{}^r$ is the orbit space under π of the part of $(E^r)^p$ projecting onto $V_0{}^p$. There is a bundle projection $E_\pi{}^r\to V_\pi$.

Certain subspaces S of the fibre of $E_\pi{}^r$ are invariant under the structural group, and under change of coordinates in M. These lead to sub-bundles E_S of $E_\pi{}^r$, which are also associated bundles. A differentiable map $f:V\to M$ induces a cross-section $f_\pi{}^r:V_\pi\to E_\pi{}^r$. By definition, f presents a self-intersection of type S at $x\in V_\pi$ if $f_\pi{}^r(x)\in E_S$. Assuming S is a submanifold of the fibre of $E_\pi{}^r$, the author states a transversality theorem: If K is a compact subset of V_π, every differentiable map of V into M can be approximated up to any order by a differentiable map $f:V\to M$ such that $f_\pi{}^r$ is transverse on E_S at every point of K, in the sense of Thom. If $f_\pi{}^r$ is transverse at every point of V_π, the set $S_f\subset V_\pi$ of self-intersections of f of type S is a submanifold of V_π, of the same codimension as that of E_S in $E_\pi{}^r$. The cohomology class of V_π dual to S_f is the image under $(f_\pi{}^r)^*$ of the class dual to E_S in $E_\pi{}^r$. If $r=0$ and $M=R^m$, this class is a characteristic class α of the p-fold covering $V_0{}^p\to V_\pi$. If $p=2$ and $\pi=Z_2$, then $\alpha=\mu^m$, where μ is the fundamental class of the double covering $V_0{}^2\to V_{Z_2}$.

Let H^* indicate cohomology mod 2, and identify $H^*(V_{Z_2})$ with the subgroup of symmetric elements of $H^*(V)\otimes H^*(V)$. Define $\Delta:H^*(V)\to H^*(V_{Z_2})$ by $\Delta(a)=a\otimes a$, and $S:H^*(V)\otimes H^*(V)\to H^*(V_{Z_2})$ by $S(a\otimes b)=a\otimes b+b\otimes a$. Theorem: The class of $H^*(V_{Z_2})$ dual to S_f can be expressed in the form $\sum_k \mu^k d_k+s$, where d_k and s

are the images, under Δ and S respectively, of classes of $H^*(V)$ and $H^*(V) \otimes H^*(V)$ belonging to the ring generated by the Stiefel-Whitney classes of V and $f^*[H^*(M)]$.

Relations between S_f and Stiefel-Whitney classes are discussed. The special case where f is an immersion is also considered. *M. W. Hirsch* (Berkeley, Calif.)

7145:

Haefliger, André. Quelques remarques sur les applications différentiables d'une surface dans le plan. Ann. Inst. Fourier. Grenoble **10** (1960), 47–60.

Let V and W be two C^∞ surfaces and let $f: V \to W$ be a C^∞ map. f will be called excellent if its singular points (points where the rank of f is <2) have neighborhoods in which f can be expressed either by $u=x$, $v=y^2$ or by $u=x$, $v=xy-y^3$, in terms of local coordinates (x, y) on V and (u, v) on W. The singular locus C is a nonsingular curve on V and singular points of the above second type are called cusps. The following results are obtained where f is excellent, $W=R^2$ and V is compact. (I) Let $p: R^3 \to R^2$ be defined by $p(u, v, w)=(u, v)$. Then there is an immersion $g: V \to R^3$ with $f=pg$ if and only if the number of cusps along each component C_i of C is even or odd according as C_i has an orientable or non-orientable neighborhood. (II) If V is orientable and C divides V into two parts V_1, V_2, then the number of cusps is not less than the difference of the Euler-Poincaré characteristics of V_1 and V_2.

The proof of (I) depends on comparing the twisting of a neighborhood of C_i with the twisting of a suitable vector field along C_i. The proof of (II) consists in considering the critical points of a certain function.

A third theorem shows that the normal degree of the boundary of a compact surface immersed in the plane is equal to the Euler-Poincaré characteristic of the surface. Again the proof depends on counting the critical points of a suitable function. Here the normal degree of a closed curve is the number of rotations of the normal as a variable point traces the whole curve.

Extensions of these theorems to varieties of dimension >2 are stated. *A. H. Wallace* (Bloomington, Ind.)

7146:

Weier, Joseph. Le caratteristiche di una trasformazione continua. Rend. Mat. e Appl. (5) **18** (1959), 11–24.

Let M be an m-dimensional euclidean manifold, N an n-dimensional differentiable manifold, $n \leq m$, f a continuous mapping from M into N. A vector field belonging to f is a continuous function v which assigns to each point p of M a vector $v(p)$ which is tangent to N at $f(p)$. Two vector fields v and w belonging to f are orthogonal if the vectors $v(p)$ and $w(p)$ are orthogonal for each point p of M. For a positive integer $r \leq n$ it is pointed out that there exist r mutually orthogonal fields w_i of vectors belonging to f such that the set of points p in M where at least one of $w_i(p)$ is the null vector is a polyhedron A of dimension $s=(m-n)+(r-1)$. If A is decomposed into oriented s-simplexes σ_i a local characteristic α_i is assigned to each σ_i and the chain $\sum \alpha_i\sigma_i$ is shown to be a cycle in a homology class which is uniquely determined by f and is termed the characteristic class of order r for f. Relations between r mutually orthogonal vector fields belonging to f and the resulting characteristic homology and cohomology classes are discussed. *P. V. Reichelderfer* (Columbus, Ohio)

7147:

Słebodziński, W. Sur les prolongements d'une connexion linéaire. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 145–150. (Russian summary, unbound insert)

Let B be a subbundle of the bundle of linear frames over a manifold V_n; it is a principal fibre bundle with group $\Gamma \subset GL(n; R)$. The author constructs in a natural way a series of principal fibre bundles $B=B_0, B_1, B_2, \cdots$ such that B_i is a bundle over B_{i-1} with group Γ_i. If $\Gamma_k=1$ for some k, then one obtains a connection in B_k canonically associated with B [cf. S. S. Chern, Colloques Internat. C.N.R.S. (Strasbourg, 1953), pp. 119–136, Centre Nat. Rech. Sci., Paris, 1953; MR **16**, 112]. *S. Kobayashi* (Vancouver, B.C.)

7148:

Ehresmann, Charles. Catégories topologiques et catégories différentiables. Colloque Géom. Diff. Globale (Bruxelles, 1958), pp. 137–150. Centre Belge Rech. Math., Louvain, 1959.

Using the notations of his previous paper [Jber. Deutsch. Math. Verein. **60** (1957), Abt. 1, 49–77; MR **20** #2392], the author defines the notion of differentiable structure and fibre space on a class. Let Φ be a category and, for each $f \in \Phi$, let $\alpha(f)$ and $\beta(f)$ be the right identity and the left identity of f, respectively. (If we consider f as a map of an object A into another object B, then $\alpha(f)=\mathrm{id}_A$ and $\beta(f)=\mathrm{id}_B$.) By a differentiable category is meant a category Φ with the structure of a differentiable manifold (not necessarily connected) such that (1) α and β are differentiable maps $\Phi \to \Phi$ of locally constant rank, (2) the composition $(g, f) \to gf$ is differentiable. Then the groupoid Π of inversible elements of Φ is open in Φ and the map $f \to f^{-1}$ is a diffeomorphism of Π. A topological category is defined in a similar manner. A fibre space over a topological category Φ is, by definition, a class S_0 with the structure of a topological space such that (1) Φ acts on S_0 (see the above cited paper for the definition of a category acting on a class), (2) the projection $p: S_0 \to \Delta$ (where Δ is the class of identities, i.e., objects of Φ) defined by (1) is continuous, (3) the map $(f, z) \to fz$ defined on a sub-class of $\Phi \times S_0$ is continuous. A differentiable fibre space over Φ is similarly defined. A category Φ is called transitive if the groupoid Π of inversible elements is transitive on Δ, i.e., for any $e, x \in \Delta$, there exists an element $l_x \in \Pi$ such that $\alpha(l_x)=e$ and $\beta(l_x)=x$. Φ is called locally trivial if, for every $x_0 \in \Delta$, there exists a lift or section σ of an open neighborhood $U(x_0)$ into Π such that $\alpha\sigma(x)=x_0$ and $\beta\sigma(x)=x$ for all x in $U(x_0)$. If S_0 is a fibre space over Φ and if Φ is transitive and locally trivial, then S_0 is a locally trivial fibre space with base Δ. If Φ is a category with the class Δ of identities (i.e., objects) and q is a map from a class B into Δ, then the induced category $q^*(\Phi)$ with the class B of identities is defined in the same way as the induced fibre space. The last section of the paper is concerned with the reduction and extension of the structure group of a fibre space (over a class). *S. Kobayashi* (Vancouver, B.C.)

7149:

Van de Ven, A. A property of algebraic varieties in complex projective spaces. Colloque Géom. Diff. Globale (Bruxelles, 1958), pp. 151–152. Centre Belge Rech. Math., Louvain, 1959.

Die algebraische Mannigfaltigkeit V_d sei singularitätenfrei in den $P_n(C)$ eingebettet. Es sei ξ ein holomorphes Teilbündel der Beschränkung des Tangentialbündels von $P_n(C)$ auf V_d. Es sei η das zum Hyperebenenschnitt von V_d gehörige Geradenbündel über V_d. Bezeichnet 1 das triviale Geradenbündel, dann ist $(\xi \oplus 1) \otimes \eta^{-1}$ ein Vektorraum-Bündel, dessen Chernsche Cohomologieklasse der komplexen Dimension i nach Multiplikation mit $(-1)^i$ durch einen algebraischen Zyklus mit nicht-negativen Vielfachheiten realisiert werden kann. Verf. formuliert seinen Satz nicht in der hier angegebenen Weise, sondern indem er die Formeln für die Chernschen Klassen von $(\xi \oplus 1) \otimes \eta^{-1}$ explizit anschreibt. Als Anwendung ergibt sich z.B., dass der $P_n(C)$ für $n \geq 2$ kein holomorphes Feld komplexer Linienelemente besitzt. *F. Hirzebruch* (Bonn)

7150:

Lichnerowicz, André. Sur les transformations analytiques d'une variété kählérienne compacte. Colloque Géom. Diff. Globale (Bruxelles, 1958), pp. 11–26. Centre Belge Rech. Math., Louvain, 1959.

The author studies the largest connected group of holomorphic transformations of a compact Kaehler manifold. In chapter I, he considers compact complex manifolds V in general and derives some consequences from the fact that $i(X)h_{(1,0)} = \text{const.}$ for any holomorphic vector field X and any holomorphic 1-form $h_{(1,0)}$. In chapter II, V is assumed to be a Kaehler manifold. He first proves the Matsushima-Yano theorem [Matsushima, Nagoya Math. J. **11** (1957), 145–150; MR **20** #995] to the effect that a 1-form ξ on V defines a holomorphic vector field if and only if $(\Delta\xi)_i = 2R_i{}^j\xi_j$ and then outlines the proof of the following theorem: if ξ is a 1-form defining a holomorphic vector field on V, then $\nabla_i\xi_j$ belongs to $S_x = \sigma_x + J\sigma_x$ for all $x \in V$, where σ_x is the holonomy algebra of V at x. The rest of the paper is concerned with various generalizations of another result of Matsushima that, for a compact Einstein-Kaehler manifold, the Lie algebra L_a of holomorphic vector fields is a direct sum (as a vector space) of the space L_1 of Killing vector fields and $(\sqrt{-1})L_1$.

S. Kobayashi (Vancouver, B.C.)

7151:

Aeppli, Alfred. Modifikation von reellen und komplexen Mannigfaltigkeiten. Comment. Math. Helv. **31** (1957), 219–301.

Un couple de variétés (V^n, S^m) est l'ensemble d'une variété V de dimension n et d'une sous-variété S de V de dimension m. Le couple est dit différentiable [resp. analytique complexe] si V, S et l'injection de S dans V sont différentiables [resp. analytiques complexes]. Une modification $\Phi : (V^n, S^m) \to (W^n, A^q)$ est un système composé de deux couples de variétés (V, S) et (W, A) et d'un homéomorphisme $\varphi' : V - S \to W - A$ tel que, pour toute suite de points p_k dans $V - S$ tendant vers S, la suite $\varphi'(p_k)$ dans W converge vers A; de plus, $n - 1 \geq \max(m, q)$; on dira que la modification "remplace A par S". La modification Φ est dite différentiable [resp. analytique complexe] si les couples et l'homéomorphisme φ' sont différentiables [resp. analytiques complexes]. Un cas particulier de modification (modification avec application, en abrégé m.a.) est obtenu ainsi : l'homéomorphisme φ' est la restriction, à $V - S$, d'une application continue φ de V sur W. Soit Φ une modification différentiable et soit $U(A)$ un voisinage ouvert de A dans W tel que le bord N de $W - U(A)$ soit une variété de dimension $n - 1$ différentiablement plongée dans W; alors N a une structure d'espace fibré en sphères de dimension $n - q - 1$ [resp. $n - m - 1$] de base A^q [resp. S^m]. Le chapitre I est consacré aux relations entre modifications (en particulier m.a.) différentiables et espaces fibrés en sphères ou possibilités de fermeture de variétés à bord ; par exemple : les modifications différentiables Φ sont, pour (W, A) donné, en correspondance biunivoque avec les fibrations différentiables de N en sphères de dimension $n - m - 1$. Si $q \leq n - 2$, la variété N étant fibrée en sphères, l'application antipodale pour chaque fibre définit une application α de N sur lui-même, d'où une fibration en 0-sphères (couples de points) et une modification Φ_α ; en particulier, si A est un point p de W, la modification Φ_α remplace p par un espace projectif réel de dimension $n - 1$ (analogue du processus σ de Hopf [Rend. Mat. e Appl. (5) **10** (1951), 169–182; MR **13**, 861]) ; si $q \geq 1$, A est remplacé par une variété fibrée par des espaces projectifs réels de dimension $n - q - 1$ (processus $\sigma^{n,q}$ réel). Les chapitres II et III sont, partiellement, valables pour des modifications "générales" dans lesquelles V et W sont des variétés homologiques orientables ou non (i.e., des polyèdres de dimension finie pour lesquels le théorème de dualité de Poincaré est valable pour un corps quelconque J ou seulement pour le corps Z_2 des entiers modulo 2), S et A étant des sous-ensembles convenables de V et W respectivement. On utilise les deux suites exactes de cohomologie, à coefficients dans J, relatives aux deux couples (V, S) et (W, A) ; ces suites sont partiellement accouplées par les isomorphismes $H^k(W, A) \to H^k(V, S)$ induits par l'homéomorphisme φ' ; la propriété : $H^k(A) = 0$ pour $k > q$ permet de déduire des propriétés de S. Exemple : Si A est un point de W et si Φ est différentiable (modification locale qui est d'ailleurs une m.a.), la variété S a un anneau de cohomologie isomorphe à celui de l'espace projectif généralisé. Des propriétés d'orientabilité des variétés (homologiques) V et S sont établies. La théorie cohomologique des espaces fibrés en sphères est utilisée avec la méthode ci-dessus dans le cas des modifications remplaçant une sous-variété de W. Dans le chapitre III relatif aux m.a., les suites exactes de (V, S) et de (W, A), accouplées par les homomorphismes induits par l'application φ, fournissent des résultats plus précis qu'au chapitre II ; voici l'un d'eux : avec des hypothèses convenables sur les dimensions, la variété S a la même structure cohomologique additive que le produit topologique de A et d'un espace projectif généralisé. Le chapitre IV est consacré aux m.a. analytiques complexes : dans le cas des variétés kählériennes compactes, les relations entre espaces vectoriels de cohomologie établies aux chapitres II et III se transportent aux espaces de formes harmoniques de types déterminés ; on en déduit, en particulier, l'invariance, par une m.a., de la dimension de l'espace vectoriel des formes holomorphes fermées de degré donné.

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7152:

Aeppli, Alfred. Reguläre Modifikation komplexer Mannigfaltigkeiten. Regulär verzweigte Überlagerungen. Comment. Math. Helv. **33** (1959), 1–22.

Les notations sont celles de #7151. On désigne par $V^{(n)}$ une variété analytique complexe (en abrégé a.c.) de dimension complexe n. Deux modifications a.c. $\Phi_i : (V_i, S_i)$

$\to(W, A)$ ($i=1, 2$) engendrées par les applications a.c. $\varphi_i: V_i \to W$ sont équivalentes s'il existe un isomorphisme $\theta: V_1 \to V_2$ tel que $\varphi_1=\varphi_2\theta$. Une modification a.c. $\Phi:(V^{(n)}, S) \to (W, A^{(q)})$ est dite régulière si elle est engendrée par une application φ a.c. et si les injections de S et A, dans V et W respectivement, sont régulières. Théorème 1: Une telle modification est équivalente à un processus $\sigma^{n,q}$ (i.e., à une modification telle que S soit un espace fibré a.c. de dimension complexe $(n-1)$, de fibre un espace projectif complexe, de base $A^{(q)}$). La conclusion subsiste si l'on suppose seulement que S est une variété de dimension $m \leqq 2n-1$, immergée sans singularité dans V. Cela généralise le théorème d'unicité relatif au processus σ dans le cas $n=2$ [H. Hopf, mêmes Comment. **29** (1955), 132–156; MR **16**, 813]. Applications: la dilatation (ou transformation monoïdale) d'une variété algébrique W le long d'une sous-variété algébrique sans singularité A est un processus $\sigma^{n,q}$; la transformée d'une variété kählérienne compacte par un processus $\sigma^{n,q}$ est kählérienne (théorème de A. Blanchard [Ann. Sci. École Norm. Sup. (3) **73** (1956), 157–202; MR **19**, 316]). Supposons maintenant que $V-S$ soit un revêtement de $W-A$; si V, W, S, A et φ sont a.c., V est dit un revêtement ramifié régulier (en abrégé r.r.). Si, en outre, pour tout voisinage $U(S)$ de S dans V, il existe un voisinage $U(A)$ de A dans W tel que $\varphi^{-1}U(A) \subset U(S)$, et si d'autres conditions sont satisfaites, l'application φ est dite application de revêtement avec ramification régulière (en abrégé a.r.). Si $\varphi|S$ est un homéomorphisme, et s'il existe des systèmes de coordonnées locales $(x_1, \cdots, x_n)$ [resp. $(y_1, \cdots, y_n)$] au voisinage de tout point x de S [resp. $\varphi(x)$ de A] tels que S soit défini par $x_1=0$ et que φ soit définie par $y_1=x_1^r$, $y_i=x_i$ ($i=2, \cdots, n$), l'application φ est appelée un "plissement" (Windung) le long de A dans W. Théorème 2: Si V est un r.r., la dimension complexe de S est $n-1$; soit q celle de A; alors: dans un voisinage de S dans V, si $q=n-1$, l'application φ est le produit d'un plissement et d'une a.r. et, si $q \leqq n-2$, l'application φ est le produit d'une a.r. et d'un processus $\sigma^{n,q}$. Cela résulte du lemme: pour qu'il existe un plissement r-uple le long de S, dans V, il faut et il suffit que la classe caractéristique du fibré normal de S dans V soit divisible par r. La technique utilisée fait intervenir des résultats connus sur les applications a.c. (théorème de Radò généralisé [H. Cartan, Math. Ann. **125** (1952), 49–50; MR **14**, 264] et B. O. Koopman [Bull. Amer. Math. Soc. **34** (1928), 565–572]).

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AUTHOR INDEX

PART A